THE SATO-TATE LAW FOR DRINFELD MODULES

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ABSTRACT. We prove an analogue of the Sato-Tate conjecture for Drinfeld modules. Using ideas of Drinfeld, J.-K. Yu showed that Drinfeld modules satisfy some Sato-Tate law, but did not describe the actual law. More precisely, for a Drinfeld module ϕ defined over a field L, he constructs a continuous representation $\rho_\infty\colon W_L \to D^\times$ of the Weil group of L into a certain division algebra, which encodes the Sato-Tate law. When ϕ has generic characteristic and L is finitely generated, we shall describe the image of ρ_∞ up to commensurability. As an application, we give improved upper bounds for the Drinfeld module analogue of the Lang-Trotter conjecture.

1. INTRODUCTION

1.1. **Notation.** We first set some notation that will hold throughout. Let F be a global function field. Let k be its field of constants and denote by q the cardinality of k. Fix a place ∞ of F and let A be the subring consisting of those functions that are regular away from ∞. For each place λ of F, let F_λ be the completion of F at λ. Let $\operatorname{ord}_\lambda$ denote the corresponding discrete valuation on F_λ, $\mathcal{O}_\lambda$ the valuation ring, and $\mathbb{F}_\lambda$ the residue field. Let d_∞ be the degree of the extension $\mathbb{F}_\infty/k$.

For a field extension L of k, let $\overline{L}$ be a fixed algebraic closure and let L^{sep} be the separable closure of L in $\overline{L}$. We will denote the algebraic closure of k in $\overline{L}$ by $\bar{k}$. Let $\operatorname{Gal}_L = \operatorname{Gal}(L^{\mathrm{sep}}/L)$ be the absolute Galois group of L. The Weil group W_L is the subgroup of Gal_L consisting of those σ for which $\sigma|_{\bar{k}}$ is an integral power $\deg(\sigma)$ of the Frobenius automorphism $x \mapsto x^q$. The map $\deg\colon W_L \to \mathbb{Z}$ is a group homomorphism. Denote by L^{perf} the perfect closure of L in $\overline{L}$.

Let $L[\tau]$ be the twisted polynomial ring with the commutation rule $\tau \cdot a = a^q\tau$ for $a \in L$; in particular, $L[\tau]$ is non-commutative if $L \neq k$. Identifying τ with X^q, we find that $L[\tau]$ is the ring of k-linear additive polynomials in $L[X]$ where multiplication corresponds to composition of polynomials. Suppose further that L is perfect. Let $L(\!(\tau^{-1})\!)$ be the skew-field consisting of twisted Laurent series in τ^{-1} (we need L to be perfect so that $\tau^{-1}\cdot a = a^{1/q}\tau$ holds). Define the valuation $\operatorname{ord}_{\tau^{-1}}\colon L(\!(\tau^{-1})\!) \to \mathbb{Z}\cup\{+\infty\}$ by $\operatorname{ord}_{\tau^{-1}}(\sum_i a_i\tau^{-i}) = \inf\{i : a_i \neq 0\}$ and $\operatorname{ord}_{\tau^{-1}}(0) = +\infty$. The valuation ring of $\operatorname{ord}_{\tau^{-1}}$ is $L[\![\tau^{-1}]\!]$, i.e., the ring of twisted formal power series in τ^{-1}.

For a ring R and a subset S, let $\operatorname{Cent}_R(S)$ be the subring of R consisting of those elements that commute with S.

1.2. **Drinfeld module background and the Sato-Tate law.** A Drinfeld module over a field L is a ring homomorphism

$$\phi\colon A \to L[\tau],\ a \mapsto \phi_a$$

such that $\phi(A)$ is not contained in the subring of constant polynomials. Let $\partial\colon L[\tau] \to L$ be the ring homomorphism $\sum_i b_i\tau^i \mapsto b_0$. The characteristic of ϕ is the kernel of $\partial\circ\phi\colon A \to L$; it is a prime ideal of A. If the characteristic of ϕ is the zero ideal, then we say that ϕ has generic characteristic. Using $\partial\circ\phi$, we shall view L as an extension of k, and as an extension of F when ϕ has generic characteristic.

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The ring $L[\tau]$ is contained in the skew field $L^{\mathrm{perf}}(\!(\tau^{-1})\!)$. The map ϕ is injective, so it extends uniquely to a homomorphism $\phi\colon F \hookrightarrow L^{\mathrm{perf}}(\!(\tau^{-1})\!)$. The function $v\colon F \to \mathbb{Z}\cup\{+\infty\}$ defined by $v(x) = \mathrm{ord}_{\tau^{-1}}(\phi_x)$ is a non-trivial discrete valuation that satisfies $v(x) \le 0$ for all non-zero $x \in A$. Therefore v is equivalent to ord_∞, and hence there exists a positive $n \in \mathbb{Q}$ that satisfies

$$\mathrm{ord}_{\tau^{-1}}(\phi_x) = n d_\infty \,\mathrm{ord}_\infty(x) \tag{1.1}$$

for all $x \in F^\times$. The number n is called the rank of ϕ and it is always an integer. Since $L^{\mathrm{perf}}(\!(\tau^{-1})\!)$ is complete with respect to $\mathrm{ord}_{\tau^{-1}}$, the map ϕ extends uniquely to a homomorphism

$$\phi\colon F_\infty \hookrightarrow L^{\mathrm{perf}}(\!(\tau^{-1})\!)$$

that satisfies (1.1) for all $x \in F_\infty^\times$. This is the starting point for the constructions of Drinfeld in [Dri77]. Let $\mathbb{F}_\infty \to L^{\mathrm{perf}}$ be the homomorphism obtained by composing $\phi|_{\mathbb{F}_\infty}$ with the map that takes an element in $L^{\mathrm{perf}}[\![\tau^{-1}]\!]$ to its constant term. So ϕ induces an embedding of $\mathbb{F}_\infty$ into L^{perf}, and hence into L itself.

Let D_ϕ be the centralizer of $\phi(A)$, equivalently of $\phi(F_\infty)$, in $\overline{L}(\!(\tau^{-1})\!)$. The ring D_ϕ is an F_∞-algebra via our extended ϕ. We shall see in §2 that D_ϕ is a central F_∞-division algebra with invariant $-1/n$. For each field extension L'/L, the ring $\mathrm{End}_{L'}(\phi)$ of endomorphisms of ϕ is the centralizer of $\phi(A)$ in $L'[\tau]$. We have inclusions $\phi(A) \subseteq \mathrm{End}_{\overline{L}}(\phi) \subseteq D_\phi$.

Following J.-K. Yu [Yu03], we shall define a continuous homomorphism

$$\rho_\infty\colon W_L \to D_\phi^\times$$

that, as we will explain, should be thought of as the Sato-Tate law for ϕ. Let us briefly describe the construction, see §2 for details. There exists an element $u \in \overline{L}(\!(\tau^{-1})\!)^\times$ with coefficients in L^{sep} such that $u^{-1}\phi(A)u \subseteq \bar{k}(\!(\tau^{-1})\!)$. For $\sigma \in W_L$, we define

$$\rho_\infty(\sigma) := \sigma(u)\tau^{\deg(\sigma)}u^{-1}$$

where σ acts on the series u by acting on its coefficients. We will verify in §2 that $\rho_\infty(\sigma)$ belongs to $D_\phi^\times$, is independent of the initial choice of u, and that ρ_∞ is indeed a continuous homomorphism. Our construction of ρ_∞ varies slightly from than that of Yu's (cf. §2.2); his representation ρ_∞ is only canonically defined up to an inner automorphism. When needed, we will make the dependence on the Drinfeld module clear by using the notation $\rho_{\phi,\infty}$ instead of ρ_∞.

Now consider a Drinfeld module $\phi\colon A \to L[\tau]$ of rank n with generic characteristic and assume that L is a finitely generated field. Choose an integral scheme X of finite type over k with function field L. For a closed point x of X, denote its residue field by $\mathbb{F}_x$. Using that A is finitely generated, we may replace X with an open subscheme such that the coefficients of all elements of $\phi(A) \subseteq L[\tau]$ are integral at each closed point x of X. By reducing the coefficients of ϕ, we obtain a homomorphism

$$\phi_x\colon A \to \mathbb{F}_x[\tau].$$

After replacing X by an open subscheme, we may assume further that ϕ_x is a Drinfeld module of rank n for each closed point x of X.

Let $P_{\phi,x}(T) \in A[T]$ be the characteristic polynomial of the Frobenius endomorphism $\pi_x := \tau^{[\mathbb{F}_x:k]} \in \mathrm{End}_{\mathbb{F}_x}(\phi_x)$; it is the degree n polynomial that is a power of the minimal polynomial of π_x over F. We shall see that ρ_∞ is unramified at x and that

$$P_{\phi,x}(T) = \det(TI - \rho_\infty(\mathrm{Frob}_x))$$

where we denote by $\det\colon D_\phi \to F_\infty$ the reduced norm. The representation ρ_∞ can thus be used to study the distribution of the coefficients of the polynomials $P_{\phi,x}(T)$ with respect to the ∞-adic topology. Though Yu showed that ϕ satisfies an analogue of Sato-Tate, he was unable to say

what the Sato-Tate law actually was. We shall address this by describing the image of ρ_∞ up to commensurability. We first consider the case where ϕ has no extra endomorphisms.

Theorem 1.1. *Let $\phi\colon A \to L[\tau]$ be a Drinfeld module with generic characteristic where L is a finitely generated field and assume that $\operatorname{End}_{\overline{L}}(\phi) = \phi(A)$. The group $\rho_\infty(W_L)$ is an open subgroup of finite index in $D_\phi^\times$.*

We will explain the corresponding equidistribution result in §1.4 after a brief interlude on elliptic curves in §1.3.

Now consider a general Drinfeld module $\phi\colon A \to L[\tau]$ with generic characteristic, L finitely generated, and no restriction on the endomorphism ring of ϕ. The reader may safely read ahead under the assumption that $\operatorname{End}_{\overline{L}}(\phi) = \phi(A)$ (indeed, a key step in the proof is to reduce to the case where ϕ has no extra endomorphisms).

The ring $\operatorname{End}_{\overline{L}}(\phi)$ is commutative and a projective module over A with rank $m \le n$, cf. [Dri74, p.569 Corollary]. Also, $E_\infty := \operatorname{End}_{\overline{L}}(\phi) \otimes_A F_\infty$ is a field of degree m over F_∞. Let B_ϕ be the centralizer of $\operatorname{End}_{\overline{L}}(\phi)$, equivalently of E_∞, in $\overline{L}(\!(\tau^{-1})\!)$; it is central E_∞-division algebra with invariant $-m/n$.

There is a finite separable extension L' of L for which $\operatorname{End}_{\overline{L}}(\phi) = \operatorname{End}_{L'}(\phi)$. We shall see that $\rho_\infty(W_{L'})$ commutes with $\operatorname{End}_{L'}(\phi)$, and hence $\rho_\infty(W_{L'})$ is a subgroup of $B_\phi^\times$. The following generalization of Theorem 1.1 says that after this constraint it taken into account, the image of ρ_∞ is, up to finite index, as large as possible.

Theorem 1.2. *Let $\phi\colon A \to L[\tau]$ be a Drinfeld module with generic characteristic where L is a finitely generated field. The group $\rho_\infty(W_L) \cap B_\phi^\times$ is an open subgroup of finite index in $B_\phi^\times$. Moreover, the groups $\rho_\infty(W_L)$ and $B_\phi^\times$ are commensurable.*

These theorems address several of the questions raised by J.K. Yu in [Yu03, §4].

1.3. **Elliptic curves.** We now recall the Sato-Tate conjecture for elliptic curves over a number field. We shall present it in a manner so that the analogy with Drinfeld modules is transparent; in particular, this strengthens the analogy presented in [Yu03].

Let $\mathbf{H}$ be the real quaternions; it is a central $\mathbb{R}$-division algebra with invariant $-1/2$. We will denote the reduced norm by $\det\colon \mathbf{H} \to \mathbb{R}$. Let $\mathbf{H}_1$ be the group of quaternions of norm 1.

For a group H, we shall denote the set of conjugacy classes by $H^\sharp$. Now suppose that H is a compact topological group and let μ be the Haar measure on H normalized so that $\mu(H) = 1$. Using the natural map $f\colon H \to H^\sharp$, we give $H^\sharp$ the quotient topology. The Sato-Tate measure on $H^\sharp$ is the measure μ_{ST} for which $\mu_{\mathrm{ST}}(U) = \mu(f^{-1}(U))$ for all open subsets $U \supseteq H^\sharp$.

Fix an elliptic curve E defined over a number field L, and let S be the set of non-zero prime ideals of $\mathcal{O}_L$ for which E has bad reduction. For each non-zero prime ideal $\mathfrak{p} \notin S$ of $\mathcal{O}_L$, let $E_\mathfrak{p}$ be the elliptic curve over $\mathbb{F}_\mathfrak{p} = \mathcal{O}_L/\mathfrak{p}$ obtained by reducing E modulo $\mathfrak{p}$, and let $\pi_\mathfrak{p}$ be the Frobenius endomorphism of $E_\mathfrak{p}/\mathbb{F}_\mathfrak{p}$. The characteristic polynomial of $\pi_\mathfrak{p}$ is the polynomial $P_{E,\mathfrak{p}}(T) \in \mathbb{Q}[T]$ of degree 2 that is a power of the minimal polynomial of $\pi_\mathfrak{p}$ over $\mathbb{Q}$. We have $P_{E,\mathfrak{p}}(T) = T^2 - a_\mathfrak{p}(E)T + N(\mathfrak{p})$ where $N(\mathfrak{p})$ is the cardinality of $\mathbb{F}_\mathfrak{p}$ and $a_\mathfrak{p}(E)$ is an integer that satisfies $|a_\mathfrak{p}(E)| \le 2N(\mathfrak{p})^{1/2}$.

Suppose that E/L does not have complex multiplication, that is, $\operatorname{End}_{\overline{L}}(E) = \mathbb{Z}$. For each prime $\mathfrak{p} \notin S$, there is a unique conjugacy class $\theta_\mathfrak{p}$ of $\mathbf{H}^\times$ such that $P_{E,\mathfrak{p}}(T) = \det(TI - \theta_\mathfrak{p})$ (this uses that $a_\mathfrak{p}(E)^2 - 4N(\mathfrak{p}) \le 0$). We can normalize these conjugacy classes by defining $\vartheta_\mathfrak{p} = \theta_\mathfrak{p}/\sqrt{N(\mathfrak{p})}$; it is the unique conjugacy class of $\mathbf{H}_1$ for which $\det(TI - \vartheta_\mathfrak{p}) = T^2 - (a_\mathfrak{p}(E)/\sqrt{N(\mathfrak{p})})T + 1$. The Sato-Tate conjecture for E/L predicts that the conjugacy classes $\{\vartheta_\mathfrak{p}\}_{\mathfrak{p}\notin S}$ are equidistributed in $\mathbf{H}_1^\sharp$

with respect to the Sato-Tate measure, i.e., for any continuous function $f\colon \mathbf{H}_1^\sharp \to \mathbb{C}$, we have

$$\lim_{x\to+\infty} \frac{1}{|\{\mathfrak{p} \notin S : N(\mathfrak{p}) \le x\}|} \sum_{\mathfrak{p}\notin S,\, N(\mathfrak{p})\le x} f(\vartheta_\mathfrak{p}) = \int_{\mathbf{H}_1^\sharp} f(\xi)d\mu_{\mathrm{ST}}(\xi).$$

Note that $\mathbf{H}_1$ and $\mathrm{SU}_2(\mathbb{C})$ are maximal compact subgroups of $(\mathbf{H}\otimes_\mathbb{R}\mathbb{C})^\times \cong \mathrm{GL}_2(\mathbb{C})$ and are thus conjugate. So our quaternion formulation agrees with the more familiar version dealing with conjugacy classes in $\mathrm{SU}_2(\mathbb{C})$. [The Sato-Tate conjecture has been proved in the case where L is totally real and E has non-integral j-invariant, cf. [CHT08, Tay08, HSBT10]]

Remark 1.3. The analogous case is a Drinfeld module $\phi\colon A \to L[\tau]$ with generic characteristic and rank 2 where L is a global function field. The algebra D_ϕ is a central F_∞-division algebra with invariant $-1/2$. For each place $\mathfrak{p}\neq\infty$ of good reduction, there is a unique conjugacy class $\theta_\mathfrak{p}$ of $D_\phi^\times$ such that $\det(TI-\theta_\mathfrak{p}) = P_{\phi,\mathfrak{p}}(T)$. This information is all encoded in our function ρ_∞, since $\theta_\mathfrak{p}$ is the conjugacy class containing $\rho_\infty(\mathrm{Frob}_\mathfrak{p})$. We will discuss the equidistribution law in §1.4; it will be a consequence of the function field version of the Chebotarev density theorem.

Now suppose that E/L has complex multiplication, and assume that $R := \mathrm{End}_{\overline{L}}(E)$ equals $\mathrm{End}_L(E)$. The ring R is an order in the quadratic imaginary field $K := R\otimes_\mathbb{Z}\mathbb{Q}$. For $\mathfrak{p}\notin S$, reduction of endomorphism rings modulo $\mathfrak{p}$ induces an injective homomorphism $K \hookrightarrow \mathrm{End}_{\mathbb{F}_\mathfrak{p}}(E_\mathfrak{p})\otimes_\mathbb{Z}\mathbb{Q}$ whose image contains $\pi_\mathfrak{p}$; let $\theta_\mathfrak{p}$ be the unique element of K that maps to $\pi_\mathfrak{p}$.

From the theory of complex multiplication, there is a continuous homomorphism

$$\rho_{E,\infty}\colon W_L \to (K\otimes_\mathbb{Q}\mathbb{R})^\times = (\mathrm{End}_L(E)\otimes_\mathbb{Z}\mathbb{R})^\times$$

such that $\rho_{E,\infty}(\mathrm{Frob}_\mathfrak{p}) = \theta_\mathfrak{p}$ for all $\mathfrak{p}\notin S$, where W_L is the Weil group of L; see [Gro80, Chap. 1 §8]. (Using the Weil group here is rather excessive; the image is abelian, so the representation factors through W_L^{ab} which in turn is isomorphic to the idele class group of L.) Choose an isomorphism $\mathbb{C} = K\otimes_\mathbb{Q}\mathbb{R}$. We can normalize by defining $\vartheta_\mathfrak{p} = \theta_\mathfrak{p}/\sqrt{N(\mathfrak{p})}$ which belongs to the group $\mathbf{S}$ of complex numbers with absolute value 1. Then the Sato-Tate law for E/L says that the elements $\{\vartheta_\mathfrak{p}\}_{\mathfrak{p}\notin S}$ are equidistributed in $\mathbf{S}$. This closely resembles the case where ϕ is a Drinfeld module of rank 2 and $\mathrm{End}_L(\phi)$ has rank 2 over A; we then have a continuous homomorphism $\rho_\infty\colon W_L \to B_\phi^\times = (\mathrm{End}_L(\phi)\otimes_A F_\infty)^\times$.

1.4. **Equidistribution law.** Let $\phi\colon A\to L[\tau]$ be a Drinfeld module of rank n. To ease notation, set $D = D_\phi$. Let $\mathcal{O}_D$ be the valuation ring of D with respect to the valuation $\mathrm{ord}_{\tau^{-1}}\colon D\to\mathbb{Z}\cup\{+\infty\}$. The continuous homomorphism $\rho_\infty\colon W_L\to D^\times$ induces a continuous representation

$$\widehat{\rho}_\infty\colon\ \mathrm{Gal}_L \to \widehat{D^\times}$$

where $\widehat{D^\times}$ is the profinite completion of $D^\times$.

Now suppose that L is finitely generated and that $\mathrm{End}_{\overline{L}}(\phi) = \phi(A)$ (similar remarks will hold without the assumption on the endomorphism ring). Choose a scheme X as in §1.2 and let $|X|$ be its set of closed points. For a subset $\mathcal{S}$ of $|X|$, define $F_\mathcal{S}(s) = \sum_{x\in\mathcal{S}} N(x)^{-s}$ where $N(x)$ is the cardinality of the residue field $\mathbb{F}_x$. The Dirichlet density of $\mathcal{S}$ is the value $\lim_{s\to d^+} F_\mathcal{S}(s)/F_{|X|}(s)$, assuming the limit exists, where d is the transcendence degree of L (see [Pin97, Appendix B] for details on Dirichlet density).

Let μ be the Haar measure on $H := \widehat{\rho}_\infty(\mathrm{Gal}_L)$ normalized so that $\mu(H) = 1$. Take an open subset U of H that is stable under conjugation. The Chebotarev density theorem then implies that the set

$$\{x\in|X| \,:\, \widehat{\rho}_\infty(\mathrm{Frob}_x)\subseteq U\}$$

has Dirichlet density $\mu(U)$, cf. [Yu03, Corollary 3.5]. This equidistribution law can be viewed as the analogue of Sato-Tate. The choice of X is not important since different choices will agree away from a set of points with density 0.

Theorem 1.1 implies that the group H is an open subgroup of $\widehat{D^\times}$. So for a "random" $x \in |X|$, the element $\rho_\infty(\mathrm{Frob}_x)$ will resemble a random conjugacy class of H, and hence a rather generic element of $\widehat{D^\times}$.

Fix a closed subgroup V of $F_\infty^\times$ that does not lie in $\mathcal{O}_\infty^\times$. That V is unbounded in the ∞-adic topology implies that the quotient group $D^\times/V$ is compact. So as a quotient of $\widehat{\rho}_\infty$, we obtain a Galois representation $\tilde{\rho}\colon \mathrm{Gal}_L \to D^\times/V$. The image $\tilde{\rho}(\mathrm{Gal}_L)$ is thus an open subgroup of finite index in $D^\times/V$ and as above, the Chebotarev density theorem gives an equidistribution law in terms of Dirichlet density. These representations can be viewed as analogues of the normalization process described in §1.3 for non-CM elliptic curves; observe that $\mathbf{H}^\times/\mathbb{R}_{>0}$ is naturally isomorphic to $\mathbf{H}_1$ where $\mathbb{R}_{>0}$ is the group of positive real numbers.

Remark 1.4. We have used Dirichlet density instead of natural density because the finite extensions of L arising from ρ_∞ are not geometric, i.e., the field of constants will grow. Natural density can be used if one keeps in mind that $\rho_\infty(\mathrm{Gal}(L^{\mathrm{sep}}/L\bar{k})) = \rho_\infty(W_L) \cap \mathcal{O}_D^\times$.

There are many possibilities for the image of ρ_∞ and hence there are many possible Sato-Tate laws for a Drinfeld module ϕ; this contrasts with elliptic curves where there are only two expected Sato-Tate laws.

To give a concrete description of an equidistribution law, we now focus on a special case: the distribution of traces of Frobenius when ρ_∞ is surjective.

For each closed point x of X, we define the degree of x to be $\deg(x) = [\mathbb{F}_x : \mathbb{F}_\infty]$. For each integer $d \geq 1$, let $|X|_d$ be the set of degree d closed points of X. Note that $|X|_d$ is empty if d is not divisible by $[\mathbb{F}_L : \mathbb{F}_\infty]$ where $\mathbb{F}_L$ is the field of constants of L. (This notion of degree depends not only on X but on the extension L/F arising from ϕ.)

For each closed point x of X, let $a_x(\phi) \in A$ be the trace of Frobenius of ϕ at x; it is $(-1)^{n-1}$ times the coefficient of T^{n-1} in $P_{\phi,x}(T)$. We have $a_x(\phi) = \mathrm{tr}(\rho_\infty(\mathrm{Frob}_x))$ where $\mathrm{tr}\colon D \to F_\infty$ is the reduced trace map. The Drinfeld module analogue of the Hasse bound says that $\mathrm{ord}_\infty(a_x(\phi)) \geq -\deg(x)/n$, and hence $a_x(\phi)\pi^{\lfloor \deg(x)/n \rfloor}$ belongs to $\mathcal{O}_\infty$ where π is a uniformizer of F_∞.

Theorem 1.5. *Let $\phi\colon A \to L[\tau]$ be a Drinfeld module of rank $n \geq 2$ with generic characteristic where L is finitely generated. Assume that $\rho_\infty(W_L) = D_\phi^\times$.*

Let π be a uniformizer for F_∞ and let μ be the Haar measure of $\mathcal{O}_\infty$ normalized so that $\mu(\mathcal{O}_\infty) = 1$. Let $\mathcal{S}$ be the set of positive integers that are divisible by $[\mathbb{F}_L : \mathbb{F}_\infty]$. Fix a scheme X as in §1.2.

(i) *For an open subset U of $\mathcal{O}_\infty$, we have*

$$\lim_{\substack{d\in\mathcal{S},\, d\not\equiv 0 \pmod{n}\\ d\to+\infty}} \frac{\#\{x \in |X|_d : a_x(\phi)\pi^{\lfloor d/n \rfloor} \in U\}}{\#|X|_d} = \mu(U).$$

(ii) *Let ν be the measure on $\mathcal{O}_\infty$ such that if U is an open subset of one of the cosets $a + \pi\mathcal{O}_\infty$ of $\mathcal{O}_\infty$, then*

$$\nu(U) = \begin{cases} (q^{d_\infty(n-1)} - 1)/(q^{d_\infty n} - 1) \cdot \mu(U) & \text{if } U \subseteq \pi\mathcal{O}_\infty, \\ q^{d_\infty(n-1)}/(q^{d_\infty n} - 1) \cdot \mu(U) & \text{otherwise.} \end{cases}$$

For an open subset U of $\mathcal{O}_\infty$, we have

$$\lim_{\substack{d\in\mathcal{S},\, d\equiv 0 \pmod{n}\\ d\to+\infty}} \frac{\#\{x \in |X|_d : a_x(\phi)\pi^{\lfloor d/n \rfloor} \in U\}}{\#|X|_d} = \nu(U).$$

Remark 1.6. Theorem 1.5(i) proves much of a conjecture of E.-U. Gekeler [Gek08, Conjecture 8.18]; which deals with rank 2 Drinfeld modules over $L = F = k(t)$ with $\pi = t^{-1}$. (Gekeler's assumptions are weaker than $\operatorname{End}_{\overline{L}}(\phi) = \phi(A)$ with ρ_∞ surjective).

1.5. **Application: Lang-Trotter bounds.** Let $\phi\colon A \to L[\tau]$ be a Drinfeld module of rank n with generic characteristic. For simplicity, we assume that L is a global function field and that $\operatorname{End}_{\overline{L}}(\phi) = \phi(A)$. Fix X as in §1.2.

Fix a value $a \in A$, and let $P_{\phi,a}(d)$ be the number of closed points x of X of degree d such that $a_x(\phi) = a$ (see the previous section for definitions). We will prove the following bound for $P_{\phi,a}(d)$ with our Sato-Tate law.

Theorem 1.7. *With assumption as above, we have*

$$P_{\phi,a}(d) \ll q^{d_\infty(1-1/n^2)d}$$

where the implicit constant depends only on ϕ and $\operatorname{ord}_\infty(a)$.

The most studied case is $n = 2$ which is analogous to the case of non-CM elliptic curves (see Remark 1.8). With $F = k(t)$, $A = k[t]$ and $L = F$, A.C. Cojocaru and C. David have shown that $P_{\phi,a}(d) \ll q^{(4/5)d}/d^{1/5}$ and $P_{\phi,0}(d) \ll q^{(3/4)d}$ where the implicit constant does not depend on a (this also can be proved with the Sato-Tate law). For $n = 2$, the above theorem gives $P_{\phi,a}(d) \ll q^{(3/4)d}$ for all a. For arbitrary rank $n \geq 2$, David [Dav01] proved that $P_{\phi,a}(d) \ll q^{\theta(n)d}/d$ where $\theta(n) := 1 - 1/(2n^2 + 4n)$. These bounds were proved using the λ-adic representations ($\lambda \neq \infty$) associated to ϕ.

Remark 1.8. Let E be a non-CM elliptic curve over $\mathbb{Q}$. Fix an integer a, and let $P_{E,a}(x)$ be the number of primes $p \leq x$ for which E has good reduction and $a_p(E) = a$. The Lang-Trotter conjecture says that there is a constant $C_{E,a} \geq 0$ such that $P_{E,a}(x) \sim C_{E,a} \cdot x^{1/2}/\log x$ as $x \to +\infty$; see [LT76] for heuristics and a description of the conjectural constant (if $C_{E,a} = 0$, then the asymptotic is defined to mean that $P_{E,a}(x)$ is bounded as a function of x). Under GRH, Murty, Murty and Saradha showed that $P_{E,a}(x) \ll x^{4/5}/(\log x)^{1/5}$ for $a \neq 0$ and $P_{E,0}(x) \ll x^{3/4}$ [MMS88].

Assuming a very strong form of the Sato-Tate conjecture for E (i.e., the L-series attached to symmetric powers of E have analytic continuation, functional equation and satisfy the Riemann hypothesis), V. K. Murty showed that $P_{E,a}(x) \ll x^{3/4}(\log x)^{1/2}$, see [Mur85].

Let $|X|_d$ be the set of closed points of X with degree d. We shall assume from now on that d is a positive integer divisible by $[\mathbb{F}_L : k]$ where $\mathbb{F}_L$ is the field of constants in L (otherwise, $|X|_d = \emptyset$ and $P_{\phi,a}(d) = 0$).

Let us give a crude heuristic for an upper bound of $P_{\phi,a}(d)$. Fix a point $x \in |X|_d$. By the Drinfeld module analogue of the Hasse bound, we have $\operatorname{ord}_\infty(a_x(\phi)) \geq -d/n$. The Riemann-Roch theorem then implies that $|\{f \in A : \operatorname{ord}_\infty(f) \geq -d/n\}| = q^{\lfloor d/n \rfloor d_\infty + 1 - g}$ for all sufficiently large d, where g is the genus of F. So assuming $a_x(\phi)$ is a "random" element of the set $\{f \in A : \operatorname{ord}_\infty(f) \geq -d/n\}$, we find that the "probability" that $a_x(\phi)$ equals a is $O(1/q^{d_\infty \cdot d/n})$. So we conjecture that

$$P_{\phi,a}(d) \ll \sum_{x \in |X|_d} \frac{1}{q^{d_\infty \cdot d/n}} = \#|X|_d \cdot \frac{1}{q^{d_\infty \cdot d/n}} \ll \frac{q^{d_\infty \cdot d}}{d} \frac{1}{q^{d_\infty \cdot d/n}} = \frac{q^{d_\infty(1-1/n)d}}{d}.$$

Remark 1.9. In this paper, we are only interested in upper bounds. The most optimistic analogue of the Lang-Trotter conjecture would be the following: there is a positive integer N and constants $C_{\phi,a}(d) \geq 0$ such that

$$P_{\phi,a}(d) \sim C_{\phi,a}(d) \cdot q^{d_\infty(1-1/n)d}/d$$

as $d \to +\infty$ where $C_{\phi,a}(d)$ depends only on ϕ, a and d modulo N. The Sato-Tate conjecture for ϕ would be an ingredient for an explicit description of the constant $C_{\phi,a}(d)$. (The conjecture

is in general false if we insist that $N = 1$. For rank 2 Drinfeld modules over $k(t)$ and $a = 0$, [Dav96, Theorem 1.2] suggests that N is usually 2.)

To prove Theorem 1.7, we will consider the image of ρ_∞ in the quotient $D_\phi^\times/(F_\infty^\times(1+\pi^j\mathcal{O}_D))$ where π is a uniformizer of F_∞ and $j \approx d/n^2$.

1.6. **Compatible system of representations.** Let $\phi\colon A \to L[\tau]$ be a Drinfeld module of rank n. For a non-zero ideal $\mathfrak{a}$ of A, let $\phi[\mathfrak{a}]$ be the group of $b \in \overline{L}$ such that $\phi_a(b) = 0$ for all $a \in A$ (where we identify each ϕ_a with the corresponding polynomial in $L[X]$). The group $\phi[\mathfrak{a}]$ is an $A/\mathfrak{a}$-module via ϕ and if $\mathfrak{a}$ is not divisible by the characteristic of ϕ, then $\phi[\mathfrak{a}]$ is a free $A/\mathfrak{a}$-module of rank n. For a fixed place $\lambda \neq \infty$ of F, let $\mathfrak{p}_\lambda$ be the maximal ideal of $\mathcal{O}_\lambda$. The λ-adic Tate module of ϕ is defined to be

$$T_\lambda(\phi) := \mathrm{Hom}_{A_\lambda}\Big(F_\lambda/\mathcal{O}_\lambda,\, \varinjlim_i \phi[\mathfrak{p}_\lambda^i]\Big).$$

If $\mathfrak{p}_\lambda$ is not the characteristic of ϕ, then $T_\lambda(\phi)$ is a free $\mathcal{O}_\lambda$-module of rank n. There is a natural Galois action on $T_\lambda(\phi)$ which gives a continuous homomorphism

$$\rho_\lambda\colon\ \mathrm{Gal}_L \to \mathrm{Aut}_{\mathcal{O}_\lambda}(T_\lambda(\phi)).$$

Now suppose that ϕ has generic characteristic and that L is a finitely generated. Take a scheme X as in §1.2. For a closed point x of X, let λ_x be the place of F corresponding to the characteristic of ϕ_x. For a place $\lambda \neq \lambda_x$ of F, we have

$$P_{\phi,x}(T) = \det(TI - \rho_\lambda(\mathrm{Frob}_x))$$

(for $\lambda \neq \infty$, we are using $\mathrm{Aut}_{\mathcal{O}_\lambda}(T_\lambda(\phi)) \cong \mathrm{GL}_n(\mathcal{O}_\lambda)$ and [Gos92, Theorem 3.2.3(b)]). This property is one of the reasons it makes sense to view ρ_∞ as a member of the family of compatible representations $\{\rho_\lambda\}$.

There is a natural map $\mathrm{End}_{\overline{L}}(\phi) \hookrightarrow \mathrm{End}_{\mathcal{O}_\lambda}(T_\lambda(\phi))$ and the image of ρ_λ commutes with $\mathrm{End}_L(\phi)$. We can now state the following important theorem of R. Pink; it follows from [Pin97, Theorem 0.2] which is an analogue of Serre's open image theorem for elliptic curves [Ser72]. Theorem 1.2 can thus be viewed as the analogue of this theorem for the place ∞; our proof will closely follow Pink's.

Theorem 1.10 (Pink)**.** *Let $\phi\colon A \to L[\tau]$ be a Drinfeld module with generic characteristic, and assume that the field L is finitely generated. Then for any place $\lambda \neq \infty$ of F, the image of*

$$\rho_\lambda\colon\ \mathrm{Gal}_L \to \mathrm{Aut}_{\mathcal{O}_\lambda}(T_\lambda(\phi))$$

is commensurable with $\mathrm{Cent}_{\mathrm{End}_{\mathcal{O}_\lambda}(T_\lambda(\phi))}(\mathrm{End}_{\overline{L}}(\phi))^\times$.

Example 1.11 (Explicit class field theory for rational function fields)**.** As an example of the utility of viewing ρ_∞ as a legitimate member of the family $\{\rho_\lambda\}_\lambda$, we give an explicit description of the maximal abelian extension F^{ab} in F^{sep} of the field $F = k(t)$, where k is a finite field with q elements. We will recover the description of F^{ab} of Hayes in [Hay74]. Using the ideas arising from this paper, we have given a description of F^{ab} for a general global function field F, see [Zyw11].

Let ∞ be the place of F correspond to the valuation for which $\mathrm{ord}_\infty(f) = -\deg f(t)$ for each non-zero $f \in k[t]$; the function t^{-1} is a uniformizer for $\mathcal{O}_\infty$. The ring of rational functions that are regular away from ∞ is $A = k[t]$. Let $\phi\colon A \to F[\tau]$ be the homomorphism of k-algebras that satisfies $\phi_t = t + \tau$; this is a Drinfeld module of rank 1 called the Carlitz module.

If $\mathfrak{p}$ is a *monic* irreducible polynomial of $k[t]$, then $\rho_\lambda(\mathrm{Frob}_\mathfrak{p}) = \mathfrak{p}$ for every place λ of F except for the one corresponding to $\mathfrak{p}$ (for $\lambda \neq \infty$, this follows from [Hay74, Cor. 2.5]). In particular, one finds that the image of $\rho_\infty\colon W_F \to D_\phi^\times = F_\infty^\times$ must lie in $\langle t\rangle \cdot (1 + t^{-1}\mathcal{O}_\infty)$. For $\lambda \neq \infty$, we make

the identification $\mathrm{Aut}_{\mathcal{O}_\lambda}(T_\lambda(\phi)) = \mathcal{O}_\lambda^\times$. Combining our λ-adic representations together, we obtain a single continuous homomorphism

$$\prod_\lambda \rho_\lambda \colon W_F^{\mathrm{ab}} \to \Big(\prod_{\lambda\neq\infty} \mathcal{O}_\lambda^\times\Big) \times \langle t\rangle \cdot (1+t^{-1}\mathcal{O}_\infty).$$

Let $\mathbf{A}_F^\times$ be the idele group of F. The homomorphism $(\prod_{\lambda\neq\infty}\mathcal{O}_\lambda^\times)\times\langle t\rangle\cdot(1+t^{-1}\mathcal{O}_\infty)\to\mathbf{A}_F^\times/F^\times$ obtained by composing the inclusion into $\mathbf{A}_F^\times$ with the quotient map is an isomorphism. Composing $\prod_\lambda \rho_\lambda$ with this map, we obtain a continuous homomorphism

$$\beta\colon W_F^{\mathrm{ab}} \to \mathbf{A}_F^\times/F^\times.$$

The map β embodies explicit class field theory for F. Indeed, it is an isomorphism and the homomorphism $W_F^{\mathrm{ab}} \xrightarrow{\sim} \mathbf{A}_F^\times/F^\times$, $s\mapsto \beta(s^{-1})$ is the inverse of the Artin map of class field theory! See Remark 3.5, for further details. In particular, observe that the homomorphism β does not depend on our choice of ∞ and ϕ.

By taking profinite completions, we obtain an isomorphism

$$\mathrm{Gal}(F^{\mathrm{ab}}/F) \xrightarrow{\sim} \Big(\prod_{\lambda\neq\infty} \mathcal{O}_\lambda^\times\Big) \times \widehat{\langle t\rangle} \cdot (1+t^{-1}\mathcal{O}_\infty).$$

of profinite groups. This isomorphism allows us to view F^{ab} as the compositum of three linearly disjoint fields. The first is the union K_1 of the fields $F(\phi[\mathfrak{a}])$ where $\mathfrak{a}$ varies over the non-zero ideals of A, see [Hay74] for details; these extensions were first constructed by Carlitz. We have $\mathrm{Gal}(K_1/F)\cong\prod_{\lambda\neq\infty}\mathcal{O}_\lambda^\times$. The second extension is the the field $K_2=\bar{k}(t)$; it satisfies $\mathrm{Gal}(K_2/F)\cong \mathrm{Gal}(\bar{k}/k)\cong\widehat{\mathbb{Z}}$.

Finally, let us describe the third field $K_3\subset F^{\mathrm{ab}}$, i.e., the subfield for which ρ_∞ induces an isomorphism $\mathrm{Gal}(K_3/F)\xrightarrow{\sim} 1+t^{-1}\mathcal{O}_\infty$. We first find a series $u=\sum_{i=0}^\infty a_i\tau^{-i}\in \overline{F}[\![\tau^{-1}]\!]^\times$ for which $u^{-1}\phi_t u=\tau$, and hence $u^{-1}\phi(A)u\subseteq \bar{k}(\!(\tau^{-1})\!)$. Expanding out $\phi_t u = u\tau$, this translates into the equations:

$$a_0\in k^\times \quad\text{ and }\quad a_{j+1}^q - a_{j+1} = -t a_j \quad \text{ for } j\geq 0.$$

Set $a_0=1$ and recursively find $a_j\in F^{\mathrm{sep}}$ that satisfy these equations. We then have a chain of fields $F=F(a_0)\subseteq F(a_1)\subseteq F(a_2)\subseteq\ldots$. Note that the field $F(a_j)$ does not depend on the choice of a_j and $[F(a_j):F]\leq q^j$. For each $j\geq 0$, let L_j be the subfield of K_3 for which ρ_∞ induces an isomorphism $\mathrm{Gal}(L_j/F)\xrightarrow{\sim}(1+t^{-1}\mathcal{O}_\infty)/(1+t^{-(j+1)}\mathcal{O}_\infty)$. The field L_j depends only on u $\pmod{\tau^{-(j+1)}\overline{F}[\![\tau^{-1}]\!]}$, so we have $L_j\subseteq F(a_j)$. Since $q^j=[L_j:F]\leq[F(a_j):F]\leq q^j$, we deduce that

$$K_3=\bigcup_{j\geq 0} F(a_j) \quad\text{ and }\quad \mathrm{Gal}(F(a_j)/F)\cong(1+t^{-1}\mathcal{O}_\infty)/(1+t^{-(j+1)}\mathcal{O}_\infty).$$

In [Hay74], Hayes constructs the three fields K_1, K_2, K_3 and then showed that their compositum is F^{ab}. The field K_3 is constructed by consider the torsion points of another Drinfeld module but starting with the ring $k[t^{-1}]$ instead. The advantage of including ρ_∞ is that the proof is easier and that the fields K_2 and K_3 arise naturally from our canonical map β.

1.7. **Overview.** In §2, we shall define our Sato-Tate representation ρ_∞ and prove its basic properties.

In §3, we prove the rank 1 case of Theorem 1.2. The proof essentially boils down to an application of class field theory. The rank 1 case will also be a key ingredient in the general proof of Theorem 1.2.

In §4, we shall prove an ∞-adic version of the Tate conjecture. The prove entails replacing ϕ with its associated A-motive (though we will not use that terminology), and then using Tamagawa's analogue of the Tate conjecture. We have avoided the temptation to define a Sato-Tate law for

general A-motives, but the author hopes to return to it in future work (the corresponding openness theorem would likely be extremely difficult for a general A-motive since the general version of Theorem 1.10 remains open).

In §5, we prove Theorem 1.1. Our proof uses most of the ingredients from Pink's proof of Theorem 1.10. In §6, we deduce Theorem 1.2 from Theorem 1.1.

Finally, in §7 and §8, we prove Theorem 1.5 and Theorem 1.7, respectively. A key ingredient in both proofs is the Chebotarev density theorem.

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2. Construction of ρ_∞

Let $\phi\colon A \to L[\tau]$ be a Drinfeld module. As noted in §1.2, ϕ extends uniquely to a homomorphism

$$\phi\colon F_\infty \hookrightarrow L^{\mathrm{perf}}(\!(\tau^{-1})\!)$$

that satisfies (1.1) for all non-zero $x \in F_\infty$.

Our first task is to prove that there exists a series $u \in \overline{L}(\!(\tau^{-1})\!)^\times$ for which $u^{-1}\phi(F_\infty)u \subseteq \bar{k}(\!(\tau^{-1})\!)$; this is shown in [Yu03, §2], but we will reprove it in order to observe that the coefficients of u actually lie in L^{sep}. Fix a non-constant $y \in A$. We have $\phi_y = \sum_{j=0}^h b_j\tau^j$ with $b_j \in L$ and $b_h \neq 0$, where $h := -nd_\infty \operatorname{ord}_\infty(y)$. Choose a solution $\delta \in L^{\mathrm{sep}}$ of $\delta^{q^h-1} = 1/b_h$. Set $a_0 = 1$ and recursively solve for $a_1, a_2, a_3 \ldots \in \overline{L}$ by the equation

$$a_i^{q^h} - a_i = -\sum_{\substack{0\le j\le h-1\\ i+j-h\geq 0}} \delta^{q^j-1} b_j a_{i+j-h}^{q^j}. \tag{2.1}$$

The a_i belong to L^{sep} since (2.1) is a separable polynomial in a_i and the values b_j and δ belong to L^{sep}.

Lemma 2.1. *With δ and a_i as above, the series $u := \delta(\sum_{i=0}^\infty a_i\tau^{-i}) \in \overline{L}(\!(\tau^{-1})\!)^\times$ has coefficients in L^{sep} and satisfies $u^{-1}\phi(A)u \subseteq \bar{k}(\!(\tau^{-1})\!)$.*

Proof. Expanding out the series $\phi_y u$ and $u\tau^h$ and comparing, we find that $\phi_y u = u\tau^h$ (use (2.1) and $\delta^{q^h-1} = 1/b_h$). Let k_h be the degree h extension of k in $\bar{k}$. The elements of the ring $\overline{L}(\!(\tau^{-1})\!)$ that commute with τ^h are $k_h(\!(\tau^{-1})\!)$. Since $\tau^h = u^{-1}\phi_y u$ belongs to the commutative ring $u^{-1}\phi(F_\infty)u$, we find that $u^{-1}\phi(F_\infty)u$ is a subset of $k_h(\!(\tau^{-1})\!)$. Thus $u \in \overline{L}(\!(\tau^{-1})\!)^\times$ has coefficients in L^{sep} and satisfies $u^{-1}\phi(F_\infty)u \subseteq \bar{k}(\!(\tau^{-1})\!)$. ☐

Choose any series $u \in \overline{L}(\!(\tau^{-1})\!)^\times$ that satisfies $u^{-1}\phi(A)u \subseteq \bar{k}(\!(\tau^{-1})\!)$ and has coefficients in L^{sep}. Define the function

$$\rho_\infty\colon W_L \to D_\phi^\times, \quad \sigma \mapsto \sigma(u)\tau^{\deg(\sigma)}u^{-1}.$$

Recall that D_ϕ is the centralizer of $\phi(A)$ in $\overline{L}(\!(\tau^{-1})\!)$. The following lemma gives some basic properties of ρ_∞; we will give the proof in §2.1. In particular, ρ_∞ is a well-defined continuous homomorphism that does not depend on the initial choice of u. Our construction varies slightly from Yu's, cf. §2.2.

Lemma 2.2.

(i) *There is a series $u \in \overline{L}(\!(\tau^{-1})\!)^\times$ that satisfies $u^{-1}\phi(F_\infty)u \subseteq \bar{k}(\!(\tau^{-1})\!)$, and any such u has coefficients in L^{sep}.*

(ii) *The ring D_ϕ is a central F_∞-division algebra with invariant $-1/n$.*

(iii) *Fix u as in (i) and take any $\sigma \in W_L$. The series $\sigma(u)\tau^{\deg(\sigma)}u^{-1}$ belongs to $D_\phi^\times$ and does not depend on the initial choice of u.*

(iv) *For* $\sigma \in W_L$, *we have* $\operatorname{ord}_{\tau^{-1}} \rho_\infty(\sigma) = -\deg(\sigma)$.

(v) *The map* $\rho_\infty\colon W_L \to D_\phi^\times$ *is a continuous group homomorphism.*

(vi) *The group* $\rho_\infty(W_L)$ *commutes with* $\operatorname{End}_L(\phi)$.

Lemma 2.3. *Assume that ϕ has generic characteristic, L is finitely generated, and let X be a scheme as in §1.2. Then the homomorphism $\rho_\infty\colon W_L \to D_\phi^\times$ is unramified at each closed point of x of X and we have $P_{\phi,x}(T) = \det(TI - \rho_\infty(\operatorname{Frob}_x))$.*

Proof. These results follow from [Yu03]; they only depend on ρ_∞ up to conjugacy so we may use Yu's construction (see §2.2). Note that Lemma 3.2 of [Yu03] should use the arithmetic Frobenius instead of the geometric one; the contents of that lemma have been reproved below in Example 2.4. □

Example 2.4. Let $\phi\colon A \to L[\tau]$ be a Drinfeld module of rank n and L a finite field. The group W_L is cyclic and generated by the automorphism $\operatorname{Frob}_L\colon x \mapsto x^{|L|}$. We have $L^{\mathrm{sep}} = \bar{k}$, and hence $u := 1$ satisfies the condition of Lemma 2.2(i). Thus $\rho_\infty(\sigma) = \sigma(u)\tau^{\deg\sigma}u^{-1} = \tau^{\deg(\sigma)}$ for all $\sigma \in W_L$, and in particular, $\rho_\infty(\operatorname{Frob}_L) = \tau^{[L:k]}$. Note that $\pi := \tau^{[L:k]}$ belongs to $\operatorname{End}_L(\phi)$.

Let E be the subfield of $\operatorname{End}_L(\phi) \otimes_A F$ generated by F and π. Let $f_\phi \in F[T]$ be the minimal polynomial of π over F. The characteristic polynomial $P_\phi(T)$ of π is the degree n polynomial that is a power of $f_\phi(T)$.

By [Dri77, Prop. 2.1], $E \otimes_F F_\infty$ is a field and hence f_ϕ is also the minimal polynomial of π over F_∞. The characteristic polynomial of the F_∞-linear map $D_\phi \to D_\phi$, $a \mapsto \pi a$ is thus a power of f_ϕ. This implies that the degree n polynomial $\det(TI - \pi)$ is a power of f_ϕ, and hence equals $P_\phi(T)$. Therefore,

$$P_\phi(T) = \det(TI - \rho_\infty(\operatorname{Frob}_L)).$$

2.1. **Proof of Lemma 2.2.** Fix a uniformizer π of F_∞. There is a unique embedding $\iota\colon F_\infty \to \bar{k}(\!(\tau^{-1})\!)$ of rings that satisfies the following conditions:

- $\iota(x) = x$ for all $x \in \mathbb{F}_\infty$,
- $\iota(\pi) = \tau^{-nd_\infty}$,
- $\operatorname{ord}_{\tau^{-1}}(\iota(x)) = nd_\infty \operatorname{ord}_\infty(x)$ for all $x \in F_\infty^\times$.

Let k_{d_∞} and k_{nd_∞} be the degree d_∞ and nd_∞ extensions of k, respectively, in $\bar{k}$. We have $\iota(F_\infty) = k_{d_\infty}(\!(\tau^{-nd_\infty})\!)$. Let D_ι be the centralizer of $\iota(F_\infty)$ in $\bar{L}(\!(\tau^{-1})\!)$; it is an F_∞-algebra via ι. Using that k_{d_∞} and τ^{nd_∞} are in $\iota(F_\infty)$, we find that $D_\iota = k_{nd_\infty}(\!(\tau^{-d_\infty})\!)$. One can verify that D_ι is a central F_∞-division algebra with invariant $-1/n$.

By Lemma 2.1, there is a series $u \in \bar{L}(\!(\tau^{-1})\!)^\times$ with coefficients in L^{sep} such that $u^{-1}\phi(F_\infty)u \subseteq \bar{k}(\!(\tau^{-1})\!)$. Take any $v \in \bar{L}(\!(\tau^{-1})\!)^\times$ that satisfies $v^{-1}\phi(F_\infty)v \subseteq \bar{k}(\!(\tau^{-1})\!)$. By [Yu03, Lemma 2.3], there exist w_1 and $w_2 \in \bar{k}[\![\tau^{-1}]\!]^\times$ such that

$$\iota(x) = w_1^{-1}(u^{-1}\phi_x u)w_1 \quad \text{and} \quad \iota(x) = w_2^{-1}(v^{-1}\phi_x v)w_2$$

for all $x \in F_\infty$. So for all $x \in F_\infty$, we have $(uw_1)\iota(x)(uw_1)^{-1} = \phi_x = (vw_2)\iota(x)(vw_2)^{-1}$ and hence

$$(w_2^{-1}v^{-1}uw_1)\iota(x)(w_2^{-1}v^{-1}uw_1)^{-1} = \iota(x).$$

Therefore $w_2^{-1}v^{-1}uw_1$ belongs to $D_\iota \subseteq \bar{k}(\!(\tau^{-1})\!)$, and hence $v = uw$ for some $w \in \bar{k}(\!(\tau^{-1})\!)^\times$. The coefficients of v lie in L^{sep} since the coefficients of u lie in L^{sep} and w has coefficients in the perfect field $\bar{k} \subseteq L^{\mathrm{sep}}$. This completes the proof of (i).

We have shown that the series $g := uw_1 \in \bar{L}(\!(\tau^{-1})\!)$ satisfies $\iota(x) = g^{-1}\phi_x g$ for all $x \in F_\infty$. The map $D_\phi \to D_\iota$, $f \mapsto g^{-1}fg$ is an isomorphism of F_∞-algebras. Therefore, D_ϕ is also a central F_∞-division algebra with invariant $-1/n$; this proves (ii).

Take any $\sigma \in W_L$. Since w has coefficients in $\bar{k}$, we have $\sigma(w) = \tau^{\deg(\sigma)} w \tau^{-\deg(\sigma)}$ and hence

$$\begin{aligned}
\sigma(v)\tau^{\deg(\sigma)}v^{-1} &= \sigma(uw)\tau^{\deg(\sigma)}(uw)^{-1} \\
&= \sigma(u)\sigma(w)\tau^{\deg(\sigma)}w^{-1}u^{-1} \\
&= \sigma(u)(\tau^{\deg(\sigma)}w\tau^{-\deg(\sigma)})\tau^{\deg(\sigma)}w^{-1}u^{-1} \\
&= \sigma(u)\tau^{\deg(\sigma)}u^{-1}.
\end{aligned}$$

This proves that $\rho_\infty(\sigma) := \sigma(u)\tau^{\deg(\sigma)}u^{-1}$ is independent of the initial choice of u.

To complete the proof of (iii), we need only show that $\rho_\infty(\sigma)$ commutes with $\phi(A)$. We will now prove (vi), which says that $\rho_\infty(\sigma)$ commutes with the even larger ring $\mathrm{End}_L(\phi)$. Take any non-zero $f \in \mathrm{End}_L(\phi)$. Since f commutes with $\phi(A)$, and hence with $\phi(F_\infty)$, we have $(fu)^{-1}\phi(F_\infty)(fu) \subseteq \bar{k}(\!(\tau^{-1})\!)$. Since $\rho_\infty(\sigma)$ does not depend on the choice of u, we have

$$\rho_\infty(\sigma) = \sigma(fu)\tau^{\deg(\sigma)}(fu)^{-1} = \sigma(f)\sigma(u)\tau^{\deg(\sigma)}u^{-1}f^{-1} = \sigma(f)\rho_\infty(\sigma)f^{-1}.$$

Since f has coefficients in L, we deduce that $\rho_\infty(\sigma)f = f\rho_\infty(\sigma)$, as desired.

For part (iv), note that

$$\begin{aligned}
\mathrm{ord}_{\tau^{-1}}(\rho_\infty(\sigma)) &= \mathrm{ord}_{\tau^{-1}}(\sigma(u)) + \mathrm{ord}_{\tau^{-1}}(\tau^{\deg(\sigma)}) - \mathrm{ord}_{\tau^{-1}}(u) \\
&= \mathrm{ord}_{\tau^{-1}}(u) - \deg(\sigma) - \mathrm{ord}_{\tau^{-1}}(u) = -\deg(\sigma).
\end{aligned}$$

It remains to prove part (v). We first show that ρ_∞ is a group homomorphism. For $\sigma_1, \sigma_2 \in W_L$, we have

$$\rho_\infty(\sigma_1\sigma_2) = (\sigma_1\sigma_2)(u)\tau^{\deg(\sigma_1\sigma_2)}u^{-1} = \sigma_1(\sigma_2(u))\tau^{\deg(\sigma_1)}\sigma_2(u)^{-1}\cdot\sigma_2(u)\tau^{\deg(\sigma_2)}u^{-1} = \rho_\infty(\sigma_1)\rho_\infty(\sigma_2).$$

We have used part (iii) along with the observation that if $u^{-1}\phi(A)u \subseteq \bar{k}(\!(\tau^{-1})\!)$, then $\sigma_2(u)^{-1}\phi(A)\sigma_2(u) \subseteq \bar{k}(\!(\tau^{-1})\!)$.

Finally, we prove that ρ_∞ is continuous. By Lemma 2.1, we may assume that u is of the form $\sum_{i=0}^\infty a_i\tau^{-i}$ with $a_0 \neq 0$. Let $\mathcal{O}_{D_\phi}$ be the valuation ring of $\mathrm{ord}_{\tau^{-1}}\colon D_\phi \to \mathbb{Z}\cup\{+\infty\}$; it is a local ring. By part (iv), we need only show that the homomorphism $\mathrm{Gal}(L^{\mathrm{sep}}/L\bar{k}) \xrightarrow{\rho_\infty} \mathcal{O}_{D_\phi}^\times$ is continuous. It thus suffices to show that for each $j \geq 1$, the homomorphism

$$\beta_j\colon \mathrm{Gal}(L^{\mathrm{sep}}/L\bar{k}) \xrightarrow{\rho_\infty} \mathcal{O}_{D_\phi}^\times \to (\mathcal{O}_{D_\phi}/\pi^j\mathcal{O}_{D_\phi})^\times$$

has open kernel, where π is a fixed uniformizer of F_∞. For each $\sigma \in \mathrm{Gal}(L^{\mathrm{sep}}/L\bar{k})$, we have $\rho_\infty(\sigma) = \sigma(u)u^{-1}$. One can check that $\beta_j(\sigma) = 1$, equivalently $\mathrm{ord}_{\tau^{-1}}(\rho_\infty(\sigma) - 1) \geq \mathrm{ord}_{\tau^{-1}}(\phi_\pi^j) = nd_\infty j$, if and only if $\mathrm{ord}_{\tau^{-1}}(\sigma(u)u^{-1} - 1) = \mathrm{ord}_{\tau^{-1}}(\sigma(u) - u)$ is at least $nd_\infty j$. Thus the kernel of β_j is $\mathrm{Gal}(F^{\mathrm{sep}}/L_j)$ where L_j is the finite extension of $L\bar{k}$ generated by the set $\{a_i\}_{0\leq i<nd_\infty j}$.

2.2. **Yu's construction.** Let us relate our representation ρ_∞ to that given by J.K. Yu in [Yu03]. Assume that L is perfect. Let $\iota\colon F_\infty \to \bar{k}(\!(\tau^{-1})\!)$ be the embedding of §2.1. Choose a series $u_0 \in \bar{L}(\!(\tau^{-1})\!)^\times$ for which $\iota(x) = u_0\phi_x u_0^{-1}$ for all $x \in F_\infty$. The representation defined in [Yu03, §2.5] is

$$\varrho_\infty\colon W_L \to D_\iota^\times, \quad \sigma \mapsto u_0\sigma(u_0)^{-1}\tau^{\deg(\sigma)}$$

where D_ι is the central F_∞-division algebra with invariant $-1/n$ described at the beginning of §2.1. The connection with our representation is that

$$\rho_\infty(\sigma) = \sigma(u_0^{-1})\tau^{\deg(\sigma)}(u_0^{-1})^{-1} = \sigma(u_0)^{-1}\tau^{\deg(\sigma)}u_0 = u_0^{-1}\varrho_\infty(\sigma)u_0$$

for all $\sigma \in W_L$. A different choice of u_0 will change ϱ_∞ by an inner automorphism of $D_\iota^\times$. (For L not perfect, one can construct $\rho_\infty\colon W_{L^{\mathrm{perf}}} \to D_\iota^\times$ as above, and then use the natural isomorphism $W_L = W_{L^{\mathrm{perf}}}$.)

2.3. **Aside: Formal modules.** Let us quickly express the above construction in terms of *formal modules*; this will not be needed elsewhere. Let $\phi\colon A \to L[\tau]$ be a Drinfeld module of rank n and assume that L is perfect. Then ϕ extends uniquely to a homomorphism $\phi\colon F_\infty \hookrightarrow L(\!(\tau^{-1})\!)$ that satisfies (1.1) for all non-zero $x \in F_\infty$. In particular, restricting to $\mathcal{O}_\infty$ defines a homomorphism $\mathcal{O}_\infty \to L[\![\tau^{-1}]\!]$.

To each formal sum $f = \sum_{i\in\mathbb{Z}} a_i\tau^i$ with $a_i \in L$, we define its adjoint by $f^* = \sum_{i\in\mathbb{Z}} a_i^{1/q^i}\tau^{-i}$. For $f_1, f_2 \in L[\![\tau^{-1}]\!]$, we have $(f_1f_2)^* = f_2^*f_1^*$ and $(f_1^*)^* = f_1$. Define the map

$$\varphi\colon \mathcal{O}_\infty \to L[\![\tau]\!], \quad x \mapsto \phi_x^*.$$

Using that $\mathcal{O}_\infty$ is commutative, we find that φ is a homomorphism that satisfies

$$\operatorname{ord}_\tau \varphi(x) = nd_\infty \operatorname{ord}_\infty(x)$$

for all $x \in \mathcal{O}_\infty$. In the language of [Dri74, §1D], φ is a formal $\mathcal{O}_\infty$-module of height n.

If one fixes a formal $\mathcal{O}_\infty$-module $\iota\colon \mathcal{O}_\infty \to \bar{k}[\![\tau]\!]$, then by [Dri74, Prop. 1.7(1)] there is a $v \in \overline{L}[\![\tau]\!]^\times$ such that $v^{-1}\varphi(x)v = \iota(x)$ for $x \in \mathcal{O}_\infty$. Let $\mathcal{D}_\varphi$ be the centralizer of $\varphi(\mathcal{O}_\infty)$ in $\overline{L}(\!(\tau)\!)$. By [Dri74, Prop. 1.7(2)], $\mathcal{D}_\varphi$ is a central F_∞-division algebra with invariant $1/n$ and $\mathcal{D}_\varphi \cap \overline{L}[\![\tau]\!]$ is the ring of integers of $\mathcal{D}_\varphi$. One can then define a continuous homomorphism

$$\varrho\colon W_L \to \mathcal{D}_\varphi^\times, \quad \sigma \mapsto \sigma(v)\tau^{\deg(\sigma)}v^{-1}.$$

For $\sigma \in W_L$, we have $\rho_\infty(\sigma) = (\varrho(\sigma)^*)^{-1}$. Note that this construction works for any formal $\mathcal{O}_\infty$-module $\mathcal{O}_\infty \to L[\![\tau]\!]$ with height $1 \le n < \infty$.

3. Drinfeld modules of rank 1

Let $\phi\colon A \to L[\tau]$ be a Drinfeld module of rank 1 with generic characteristic. For a place $\lambda \neq \infty$ of F, the Tate module $T_\lambda(\phi)$ is a free $\mathcal{O}_\lambda$-module of rank 1. The Galois action on $T_\lambda(\phi)$ commutes with the $\mathcal{O}_\lambda$-action, and hence our Galois representation ρ_λ is of the form

$$\rho_\lambda\colon \operatorname{Gal}_L \to \operatorname{Aut}_{\mathcal{O}_\lambda}(T_\lambda(\phi)) = \mathcal{O}_\lambda^\times.$$

For the place ∞, we have defined a representation

$$\rho_\infty\colon W_L \to D_\phi^\times = F_\infty^\times,$$

where D_ϕ equals F_∞ since it is a central F_∞-division algebra with invariant -1. In this section, we will prove the following proposition, whose corollary is the rank 1 case of Theorem 1.2.

Proposition 3.1. *Let $\phi\colon A \to L[\tau]$ be a Drinfeld module of rank 1 with generic characteristic and assume that L is a finitely generated field. Then the group $\big(\prod_\lambda \rho_\lambda\big)(W_L)$ is an open subgroup with finite index in $\big(\prod_{\lambda\neq\infty} \mathcal{O}_\lambda^\times\big) \times F_\infty^\times$.*

Corollary 3.2. *Let $\phi\colon A \to L[\tau]$ be a Drinfeld module of rank 1 with generic characteristic and assume that L is a finitely generated field. Then $\rho_\infty(W_L)$ is an open subgroup with finite index in $D_\phi^\times = F_\infty^\times$.*

3.1. **Proof of Proposition 3.1.** Since ϕ has generic characteristic, it induces an embedding $F \to L$ that we view as an inclusion. The following lemma allows us to reduce to the case where L is a global function field and L/F is an abelian extension.

Lemma 3.3. *If Proposition 3.1 holds in the special case where L is a finite separable abelian extension of F, then the full proposition holds.*

Proof. Let H_A be the maximal unramified abelian extension of F in F^{sep} for which the place ∞ splits completely; it is a finite abelian extension of F. Choose an embedding $H_A \subseteq L^{\mathrm{sep}}$. By [Hay92, §15] (and our generic characteristic and rank 1 assumptions on ϕ), there is a Drinfeld module $\phi'\colon A \to H_A[\tau]$ such that ϕ and ϕ' are isomorphic over L^{sep} (since L is a finitely generated extension of F, we can choose an embedding of L into the field $\mathbf{C}$ of loc. cit.). Moreover, there is a finite extension L' of LH_A such that ϕ and ϕ' are isomorphic over L'. Therefore, $(\prod_\lambda \rho_{\phi,\lambda})(W_{L'})$ and $(\prod_\lambda \rho_{\phi',\lambda})(W_{L'})$ are equal in $(\prod_{\lambda\neq\infty} \mathcal{O}_\lambda^\times) \times F_\infty^\times$.

By the hypothesis of the lemma, we may assume that $(\prod_\lambda \rho_{\phi',\lambda})(W_{H_A})$ is an open subgroup of finite index in $(\prod_{\lambda\neq\infty} \mathcal{O}_\lambda^\times)\times F_\infty^\times$. Replacing H_A by the finitely generated extension L', we find that $(\prod_\lambda \rho_{\phi',\lambda})(W_{L'})$ is still an open subgroup of finite index in $(\prod_{\lambda\neq\infty} \mathcal{O}_\lambda^\times) \times F_\infty^\times$ (though possibly of larger index). Therefore, $(\prod_\lambda \rho_{\phi,\lambda})(W_L)$ contains $(\prod_\lambda \rho_{\phi,\lambda})(W_{L'}) = (\prod_\lambda \rho_{\phi',\lambda})(W_{L'})$ which is open and of finite index in $(\prod_{\lambda\neq\infty} \mathcal{O}_\lambda^\times) \times F_\infty^\times$. $\square$

By the above lemma, we may assume without loss of generality that L is a finite separable and abelian extension of F. The benefit of L being a global function field is that we will be able to use class field theory. Since $\rho_\lambda|_{W_L}$ is continuous with commutative image, it factors through the maximal abelian quotient W_L^{ab} of W_L. Let $\mathbf{A}_L^\times$ be the group of ideles of L. For each place λ of F, we define the continuous homomorphism

$$\widetilde{\rho}_\lambda\colon \mathbf{A}_L^\times \to W_L^{\mathrm{ab}} \xrightarrow{\rho_\lambda} F_\lambda^\times$$

where the first homomorphism is the Artin map of class field theory. The homomorphism $\widetilde{\rho}_\lambda$ is trivial on $L^\times$, and has image in $\mathcal{O}_\lambda^\times$ when $\lambda \neq \infty$. Define $L_\lambda := L \otimes_F F_\lambda$ and let $N_\lambda\colon L_\lambda \to F_\lambda$ be the corresponding norm map. Define the continuous homomorphism

$$\chi_\lambda\colon \mathbf{A}_L^\times \to F_\lambda^\times, \quad \alpha \mapsto \widetilde{\rho}_\lambda(\alpha) N_\lambda(\alpha_\lambda)$$

where α_λ is the component of α in $L_\lambda^\times = \prod_{v|\lambda} L_v^\times$.

Let S be the set of places of L for which ϕ has bad reduction or which lie over ∞. For $v \notin S$, let λ_v be the place of F lying under v. For each place $v \notin S$ of L, define $\pi_v := \rho_\infty(\mathrm{Frob}_v)$. By Lemma 2.3, π_v belongs to $F^\times$; it also equals $\rho_\lambda(\mathrm{Frob}_v)$ for all $\lambda \neq \lambda_v$. For each place $v \notin S$ of L and λ of F, we have

$$\mathrm{ord}_\lambda(\pi_v) = \begin{cases} [\mathbb{F}_v : \mathbb{F}_{\lambda_v}] & \text{if } \lambda = \lambda_v, \\ -[\mathbb{F}_v : \mathbb{F}_\infty] & \text{if } \lambda = \infty, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

cf. [Dri77, Proposition 2.1]. We now show that χ_λ is independent of λ.

Lemma 3.4. *There is a unique character $\chi\colon \mathbf{A}_L^\times \to F^\times$ that satisfies the following conditions:*

(a) $\ker(\chi)$ *is an open subgroup of* $\mathbf{A}_L^\times$.
(b) *If $\alpha \in L^\times$, then $\chi(\alpha) = N_{L/F}(\alpha)$.*
(c) *If $\alpha = (\alpha_v)$ is an idele with $\alpha_v = 1$ for $v \in S$, then $\chi(\alpha) = \prod_{v\notin S} {\pi_v}^{\mathrm{ord}_v(\alpha_v)}$.*

For every place λ of F, we have $\chi_\lambda(\alpha) = \chi(\alpha)$ for all $\alpha \in \mathbf{A}_L^\times$.

Proof. Fix a place λ of F. If $\alpha \in L^\times$, then $\chi_\lambda(\alpha) = N_\lambda(\alpha) = N_{L/F}(\alpha)$ since $\widetilde{\rho}_\lambda$ is trivial on $L^\times$. Let S_λ be those places of L that belong to S or lie over λ. For an idele $\alpha \in \mathbf{A}_L^\times$ satisfying $\alpha_v = 1$ for $v \in S_\lambda$, we have $\chi_\lambda(\alpha) = \widetilde{\rho}_\lambda(\alpha)$ which equals $\prod_{v\notin S_\lambda} \rho_\lambda(\mathrm{Frob}_v)^{\mathrm{ord}_v(\alpha_v)}$ since ρ_λ is unramified outside S_λ. Therefore, $\chi_\lambda(\alpha) = \prod_{v\notin S_\lambda} {\pi_v}^{\mathrm{ord}_v(\alpha_v)}$.

Define $U = \prod_v \mathcal{O}_v^\times$; it is an open subgroup of $\mathbf{A}_L^\times$. Consider an idele $\beta \in U$ that satisfies $\beta_v = b \in L^\times$ for all $v \in S_\lambda$. We then have

$$\chi_\lambda(\beta) = \chi_\lambda(b)\chi_\lambda(b^{-1}\beta) = N_{L/F}(b) \prod_{v \notin S_\lambda} \pi_v^{\operatorname{ord}_v(b^{-1}\beta_v)} = N_{L/F}(b) \prod_{v \notin S_\lambda} \pi_v^{-\operatorname{ord}_v(b)},$$

which is an element of $F^\times$. Take a place $\lambda' \neq \infty$ of F. By (3.1) and using that $\operatorname{ord}_v(b) = 0$ for $v \in S_\lambda$, we have

$$\begin{aligned}\operatorname{ord}_{\lambda'}\Big(\prod_{v \notin S_\lambda} \pi_v^{\operatorname{ord}_v(b)}\Big) &= \sum_{v|\lambda'} \operatorname{ord}_v(b) \operatorname{ord}_{\lambda'}(\pi_v)\\ &= \sum_{v|\lambda'} [\mathbb{F}_v : \mathbb{F}_{\lambda'}] \operatorname{ord}_v(b)\\ &= \sum_{v|\lambda'} \operatorname{ord}_{\lambda'} N_{L_v/F_{\lambda'}}(b) = \operatorname{ord}_{\lambda'} N_{L/F}(b).\end{aligned}$$

Therefore, $\operatorname{ord}_{\lambda'}(\chi_\lambda(\beta)) = 0$ for all $\lambda' \neq \infty$, and hence $\chi_\lambda(\beta)$ belongs to $A^\times = k^\times$. By weak approximation, the ideles $\beta \in U$ with $\beta_v = b \in L^\times$ for all $v \in S_\lambda$ are dense in U. Since χ_λ is continuous, we deduce that $\chi_\lambda(U) \subseteq k^\times$ and hence $\ker(\chi_\lambda)$ is an open subgroup of $\mathbf{A}_L^\times$. The group $\mathbf{A}_L^\times / \ker(\chi_\lambda)$ is generated by $L^\times$ and ideles with 1 at the places $v \in S_\lambda$, so χ_λ takes values in $F^\times$.

Define $\chi := \chi_\infty$. We have just seen that χ takes values in $F^\times$ and satisfies conditions (a), (b) and (c). Now suppose that $\chi' \colon \mathbf{A}_L^\times \to F^\times$ is a group homomorphism that satisfies the following conditions:

- $\ker(\chi')$ is an open subgroup of $\mathbf{A}_L^\times$.
- If $\alpha \in L^\times$, then $\chi'(\alpha) = N_{L/F}(\alpha)$.
- There is a finite set $S' \supseteq S$ of places of L such that $\chi'(\alpha) = \prod_{v \notin S'} {\pi_v}^{\operatorname{ord}_v(\alpha_v)}$ for all ideles α with $\alpha_v = 1$ for $v \in S'$.

The character χ' is determined by its values on the group $\mathbf{A}_L^\times/(\ker(\chi') \cap \ker(\chi))$, and this group is generated by $L^\times$ and the ideles with v-components equal to 1 for $v \in S'$. Since χ and χ' agree on such elements, we find that $\chi' = \chi$. This proves the uniqueness of a character satisfying conditions (a), (b) and (c). With $\chi' = \chi_\lambda$ and $S' = S_\lambda$, we conclude that $\chi_\lambda = \chi$. □

Let C_F and C_L be the idele class groups of F and L, respectively. The natural quotient map $\big(\prod_{\lambda \neq \infty} \mathcal{O}_\lambda^\times\big) \times F_\infty^\times \to C_F$ has kernel $k^\times$ and its image is an open subgroup of finite index in C_F. Since the $\widetilde{\rho}_\lambda$ are trivial on $L^\times$, we can define a homomorphism $f \colon C_L \to C_F$ that takes the idele class containing $\alpha \in \mathbf{A}_L$ to the idele class of C_F containing $(\widetilde{\rho}_\lambda(\alpha))_\lambda$. To prove the proposition, it suffices to show that the image of f is open with finite index in C_F. By the definition of the χ_λ and Lemma 3.4, we have

$$(\widetilde{\rho}_\lambda(\alpha))_\lambda = (\chi_\lambda(\alpha) N_\lambda(\alpha_\lambda)^{-1})_\lambda = \chi(\alpha)(N_\lambda(\alpha_\lambda)^{-1})_\lambda$$

Therefore, $f(\alpha) = N_{L/F}(\alpha)^{-1}$ for all $\alpha \in C_L$, where $N_{L/F} \colon C_L \to C_F$ is the norm map. Class field theory tells us that $N_{L/F}(C_L)$ is an open subgroup of C_F and the index $[C_F : N_{L/F}(C_L)]$ equals $[L : F]$; the same thus holds for f.

Remark 3.5. Consider the special case where $A = k[t]$, $F = k(t)$, and $\phi \colon k[t] \to F[\tau]$ is the Carlitz module of Example 1.11. As noted in Example 1.11, we have a continuous homomorphism

$$\beta \colon W_F^{\mathrm{ab}} \to \Big(\prod_{\lambda \neq \infty} \mathcal{O}_\lambda^\times\Big) \times \langle t \rangle \cdot (1 + t^{-1}\mathcal{O}_\infty) \xrightarrow{\sim} C_F$$

where the first map is $\prod_\lambda \rho_\lambda$ and the second is the quotient map. Composing β with the Artin map of F, we obtain a homomorphism $f \colon C_F \to C_F$ which from the calculation above is $f(\alpha) =$

$N_{F/F}(\alpha)^{-1} = \alpha^{-1}$. Therefore, $W_F^{\mathrm{ab}} \to C_F$, $\sigma \mapsto \beta(\sigma^{-1})$ is the inverse of the Artin map for F as claimed in Example 1.11.

4. Tate conjecture

Let $\phi\colon A \to L[\tau]$ be a Drinfeld module of rank n and let D_ϕ be the centralizer of $\phi(A)$ in $\overline{L}(\!(\tau^{-1})\!)$. Using the extended map $\phi\colon F_\infty \to L^{\mathrm{perf}}(\!(\tau^{-1})\!)$, we have shown that D_ϕ is a central F_∞-division algebra with invariant $-1/n$. In §2, we constructed a continuous representation

$$\rho_\infty\colon W_L \to D_\phi^\times.$$

We can view $\mathrm{End}_L(\phi) \otimes_A F_\infty$ as a F_∞-subalgebra of D_ϕ; it commutes with the image of ρ_∞. The following ∞-adic analogue of the Tate conjecture, says that $\mathrm{End}_L(\phi) \otimes_A F_\infty$ is precisely the centralizer of $\rho_\infty(W_L)$ in D_ϕ, at least assuming that L is finitely generated and ϕ has generic characteristic.

Theorem 4.1. *Let $\phi\colon A \to L[\tau]$ be a Drinfeld module with generic characteristic and L a finitely generated field. Then the centralizer of $\rho_\infty(W_L)$ in D_ϕ is $\mathrm{End}_L(\phi) \otimes_A F_\infty$.*

For the rest of the section, assume that L is a finitely generated field. Recall that for a place $\lambda \neq \infty$, the λ-adic version of the Tate conjecture says that the natural map

$$\mathrm{End}_L(\phi) \otimes_A F_\lambda \to \mathrm{End}_{F_\lambda[\mathrm{Gal}_L]}(V_\lambda(\phi)) \tag{4.1}$$

is an isomorphism. This is a special case of theorems proved independently by Taguchi [Tag95] and Tamagawa [Tam95]; we will make use of Tamagawa's more general formulation. We can give $\mathrm{End}_{F_\lambda}(V_\lambda(\phi))$ a Gal_L-action by $\sigma(f) := \rho_\lambda(\sigma) \circ f \circ \rho_\lambda(\sigma)^{-1}$. That (4.1) is an isomorphism is equivalent to having $\mathrm{End}_{F_\lambda}(V_\lambda(\phi))^{\mathrm{Gal}_L} = \mathrm{End}_L(\phi) \otimes_A F_\lambda$. For the ∞-adic version, the ring D_ϕ has a natural $\mathrm{Gal}(\overline{L}/L)$-action and the subring $D_\phi^{\mathrm{Gal}(\overline{L}/L)} = \mathrm{Cent}_{L^{\mathrm{perf}}(\!(\tau^{-1})\!)}(\phi(A))$ certainly contains $\mathrm{End}_L(\phi) \otimes_A F_\infty$; we will show that they are equal, and from the following lemma, deduce Theorem 4.1.

Lemma 4.2. *If $D_\phi^{\mathrm{Gal}(\overline{L}/L)} = \mathrm{End}_L(\phi) \otimes_A F_\infty$, then the centralizer of $\rho_\infty(W_L)$ in D_ϕ is $\mathrm{End}_L(\phi) \otimes_A F_\infty$.*

Proof. Fix an $f \in D_\phi$ that commutes with $\rho_\infty(W_L)$. Take any $\sigma \in W_L$. The series f and $\rho_\infty(\sigma)$ commute, so we have

$$\sigma(u)\tau^{\deg(\sigma)}u^{-1} \cdot f = f \cdot \sigma(u)\tau^{\deg(\sigma)}u^{-1},$$

where u is a series as in Lemma 2.2(i). Therefore,

$$\sigma(u)^{-1} f \sigma(u) = \tau^{\deg(\sigma)}(u^{-1}fu)\tau^{-\deg(\sigma)} = \sigma(u^{-1}fu)$$

where the last equality uses that $u^{-1}fu$ has coefficients in $\bar{k}$. Since W_L is dense in Gal_L, we have $\sigma(u)^{-1}f\sigma(u) = \sigma(u^{-1}fu)$ for all $\sigma \in \mathrm{Gal}_L$ and hence also for all $\sigma \in \mathrm{Gal}(\overline{L}/L)$. Therefore, $\sigma(u)^{-1}f\sigma(u) = \sigma(u)^{-1}\sigma(f)\sigma(u)$ and hence $\sigma(f) = f$, for all $\sigma \in \mathrm{Gal}(\overline{L}/L)$. So f belongs to $D_\phi^{\mathrm{Gal}(\overline{L}/L)}$ and is thus an element of $\mathrm{End}_L(\phi) \otimes_A F_\infty$ by assumption. This proves that the centralizer of $\rho_\infty(W_L)$ in D_ϕ is contained in $\mathrm{End}_L(\phi) \otimes_A F_\infty$.; we have already noted that the other inclusion holds. ☐

The rest of §4 is dedicated to proving Theorem 4.1. To relate our construction with the work of Tamagawa, it will be useful to replace ϕ with its corresponding A-motive. We give enough background to prove the theorem; this material will not be needed outside §4.

4.1. **Étale τ-modules.** Let L be an extension field of k (as usual, k is a fixed finite field with q elements). Let $L(\!(t^{-1})\!)$ be the (commutative) ring of Laurent series in t^{-1} with coefficients in L. Define the ring homomorphism

$$\sigma\colon L(\!(t^{-1})\!) \to L(\!(t^{-1})\!), \quad \sum_i c_i t^{-i} \mapsto \sum_i c_i^q t^{-i}.$$

Let R be a subring of $L(\!(t^{-1})\!)$ that is stable under σ; for example, $L[t]$, $L(t)$ and $L(\!(t^{-1})\!)$.

Definition 4.3. A τ-module over R is a pair (M, τ_M) consisting of an R-module M and a σ-semilinear map $\tau_M\colon M \to M$ (i.e., τ_M is additive and satisfies $\tau_M(fm) = \sigma(f)\tau_M(m)$ for all $f \in R$ and $m \in M$). A morphism of τ-modules is an R-module homomorphism that is compatible with the τ maps.

When convenient, we shall denote a τ-module (M, τ_M) simply by M. We can view R as a τ-module over itself by setting $\tau_R = \sigma|_R$. For an R-module M, denote by $\sigma^*(M)$ the scalar extension $R \otimes_{\sigma,R} M$ of M by $\sigma\colon R \to R$. Giving a σ-semilinear map $\tau_M\colon M \to M$ is thus equivalent to giving an R-linear map

$$\tau_{M,\mathrm{lin}}\colon \sigma^*(M) \to M$$

which we call the linearization of τ_M. We say that a τ-module M over R is étale if M is a free R-module of finite rank and the linearization $\tau_{M,\mathrm{lin}}\colon \sigma^*(M) \to M$ is an isomorphism.

Let M_1 and M_2 be τ-modules over R. We define $M_1 \otimes_R M_2$ to be the τ-module whose underlying R-module is $M_1 \otimes_R M_2$ with τ map determined by $\tau_{M_1\otimes_R M_2}(m_1 \otimes m_2) = \tau_{M_1}(m_1) \otimes \tau_{M_2}(m_2)$. Now suppose that M_1 is étale. Define the R-module $H := \mathrm{Hom}_R(M_1, M_2)$. Let $\tau_H\colon H \to H$ be the σ-semilinear that corresponds to the R-linear map

$$\sigma^*(H) \to H, \quad f \mapsto \tau_{M_2,\mathrm{lin}} \circ f \circ \tau_{M_1,\mathrm{lin}}^{-1},$$

where we are using the natural isomorphism $\sigma^*(H) \cong \mathrm{Hom}_R(\sigma^*(M_1), \sigma^*(M_2))$. The pair (H, τ_H) is a τ-module over R. If M_1 and M_2 are both étale over R, then H is also étale.

Suppose that $R \subseteq R'$ are subrings of $L(\!(t^{-1})\!)$ which are stable under σ. Let M be a τ-module over R. We can then give $R' \otimes_R M$ the structure of a τ-module over R'. If M is étale, then $R' \otimes_R M$ is an étale τ-module over R'.

For a τ-module M, let M^τ be the group of $m \in M$ for which $\tau_M(m) = m$; it is a module over the ring $R_0 := \{r \in R : \sigma(r) = r\}$ (for $R = L(t)$ and $L(\!(t^{-1})\!)$, we have $R_0 = k(t)$ and $k(\!(t^{-1})\!)$, respectively). Let H be the τ-module $\mathrm{Hom}_R(M_1, M_2)$ where M_1 and M_2 are τ-modules and M_1 is étale; then H^τ agrees with the set $\mathrm{Hom}(M_1, M_2)$ of endomorphisms $M_1 \to M_2$ of τ-modules.

4.2. **Weights.** Fix a separably closed extension K of k. We shall describe the étale τ-modules over $K(\!(t^{-1})\!)$; it turns out that the category of such τ-modules is semisimple, we first define the simple ones.

Definition 4.4. Let $\lambda = s/r$ be a rational number with r and s relatively prime integers and $r \geq 1$. Define the free $K(\!(t^{-1})\!)$-module

$$N_\lambda := K(\!(t^{-1})\!)e_1 \oplus \cdots \oplus K(\!(t^{-1})\!)e_r.$$

Let $\tau_\lambda\colon N_\lambda \to N_\lambda$ be the σ-semilinear map that satisfies $\tau_\lambda(e_i) = e_{i+1}$ for $1 \leq i < r$ and $\tau_\lambda(e_r) = t^s e_1$. The pair $(N_\lambda, \tau_\lambda)$ is an étale τ-module over $K(\!(t^{-1})\!)$.

Proposition 4.5.

(i) *If M is an étale τ-module over $K(\!(t^{-1})\!)$, then there are unique rational numbers $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$ such that*
- $M \cong N_{\lambda_1} \oplus \cdots \oplus N_{\lambda_m}$.

- *the t^{-1}-adic valuations of the roots of the characteristic polynomial of τ_M expressed on any $K(\!(t^{-1})\!)$-basis of M are $\{-\lambda_i\}_i$, with each λ_i counted with multiplicity $\dim N_\lambda$.*

(ii) *For $\lambda \in \mathbb{Q}$, the ring $\mathrm{End}(N_\lambda)$ is a central $k(\!(t^{-1})\!)$-division algebra with Brauer invariant λ.*

Proof. This follows from [Lau96, Appendix B]; although the proposition is only proved for a particular field K, nowhere do the proofs make use of anything stronger then K being separably closed. This was observed by Taelman in [Tae09, §5]; his notion of a "Dieudonné t-module" corresponds with étale τ-modules over $K(\!(t^{-1})\!)$ (in Definition 5.1.1 of loc. cit. one should have $K(\!(t^{-1})\!)\sigma(V) = V$). $\square$

We call the rational numbers λ_i of Proposition 4.5(i) the weights of M. If all the weights of M equal λ, then we say that M is pure of weight λ.

Lemma 4.6. [Tae09, Prop. 5.14] *Fix a rational number $\lambda = s/r$ with r and s relatively prime integers and $r \geq 1$. Let M be an étale τ-module over $K(\!(t^{-1})\!)$ with K an algebraically closed extension of k. The following are equivalent:*

- *M is pure of weight λ.*
- *there exists a $K[\![t^{-1}]\!]$-lattice $\Lambda \subseteq M$ such that $\tau_M^r(\Lambda) = t^s\Lambda$.*

Let L be a field extension of k (not necessarily separably closed) and let M be an étale τ-module over $L(t)$. The weights of M are the weights of the τ-module $K(\!(t^{-1})\!) \otimes_{L(t)} M$ over $K(\!(t^{-1})\!)$ where K is any separably closed field containing L. Again, we say that M is pure of weight λ if all the weights of M equal λ. We now give a criterion for M to be pure of weight 0.

Lemma 4.7. *Define the subring $\mathcal{O} := L[t^{-1}]_{(t^{-1})} = L(t) \cap L[\![t^{-1}]\!]$ of $L(t)$; it is a local ring with quotient field $L(t)$. Let M be an étale τ-module over $L(t)$. Then the following are equivalent:*

(a) *M is pure of weight 0.*

(b) *There is an $\mathcal{O}$-submodule N of M stable under τ_M such that $(N, \tau_M|_N)$ is an étale τ-module over $\mathcal{O}$ and the natural map $L(t) \otimes_{\mathcal{O}} N \to M$ of τ-modules is an isomorphism.*

Proof. First suppose that M is pure of weight 0. By Lemma 4.6, there is an $\overline{L}[\![t^{-1}]\!]$-lattice Λ of $\overline{L}(\!(t^{-1})\!) \otimes_{L(t)} M$ such that $\tau_M(\Lambda) = \Lambda$. Fix a basis $e_1, \ldots, e_d$ of M over $L(t)$; we may assume that the e_i are contained in Λ. Let N be the $\mathcal{O}$-submodule of M generated by the set $\mathcal{B} = \{\tau_M^j(e_i) : 1 \leq i \leq d,\, j \geq 1\}$. We can write each $v \in \mathcal{B}$, uniquely in the form $v = \sum_i a_i e_i$ with $a_i \in L(t)$; let α be the infimum of the values $\mathrm{ord}_{t^{-1}}(a_i)$ over all $i \in \{1 \ldots, d\}$ and $v \in \mathcal{B}$. We find that α is finite, since N is contained in the $\overline{L}[\![t^{-1}]\!]$-lattice Λ which is stable under τ_M. Using that α is finite, we find that N is a free $\mathcal{O}$-module of rank d which is stable under τ_M and that the map $L(t) \otimes_{\mathcal{O}} N \to M$ is an isomorphism. The τ-module $(N, \tau_M|N)$ is étale since (M, τ_M) is étale. It is now clear that N satisfies all the conditions of (b).

Now suppose there is an $\mathcal{O}$-submodule N satisfying the conditions of (b). Then $\Lambda := \overline{L}[\![t^{-1}]\!] \otimes_{\mathcal{O}} N$ is a $\overline{L}[\![t^{-1}]\!]$-lattice in $\overline{L}(\!(t^{-1})\!) \otimes_{L(t)} M$ that satisfies $\tau_{\overline{L}(\!(t^{-1})\!) \otimes_{L(t)} M}(\Lambda) = \Lambda$. Lemma 4.6 implies that $\overline{L}(\!(t^{-1})\!) \otimes_{L(t)} M$, and hence M also, is pure of weight 0. $\square$

4.3. **Tate conjecture.** Let M be an étale τ-module over $L(t)$. The group Gal_L acts on $M' := L^{\mathrm{sep}}(\!(t^{-1})\!) \otimes_{L(t)} M$ via its action on the coefficients of $L^{\mathrm{sep}}(\!(t^{-1})\!)$. The Gal_L-action on M' commutes with $\tau_{M'}$, so M'^{τ} is a vector space over $k(\!(t^{-1})\!)$ with an action of Gal_L.

Theorem 4.8. *Let L be a finitely generated extension of k. Let M be an étale τ-module over $L(t)$ that is pure of weight 0. Then the natural map*

$$M^\tau \otimes_{k(t)} k(\!(t^{-1})\!) \to \big((L^{\mathrm{sep}}(\!(t^{-1})\!) \otimes_{L(t)} M)^\tau\big)^{\mathrm{Gal}_L}$$

is an isomorphism of finite dimensional $k(\!(t^{-1})\!)$-vector spaces.

Proof. For an étale τ-module M over $L(t)$, we define

$$\widehat{V}(M) := (L^{\mathrm{sep}}(\!(t^{-1})\!) \otimes_{L(t)} M)^{\tau};$$

it is a $k(\!(t^{-1})\!)$-vector space with a natural action of Gal_L. Let M' and M be étale τ-modules over $L(t)$ that are pure of weight 0. There is a natural homomorphism

$$\mathrm{Hom}(M', M) \otimes_{k(t)} k(\!(t^{-1})\!) \to \mathrm{Hom}_{k(\!(t^{-1})\!)[\mathrm{Gal}_L]}\big(\widehat{V}(M'), \widehat{V}(M)\big) \tag{4.2}$$

of vector spaces over $k(\!(t^{-1})\!)$. We claim that (4.2) is an isomorphism. In the notation of Tamagawa in [Tam95], M' and M are "restricted $L(t)\{\tau\}$-modules that are étale at $t^{-1} = 0$". That M is an étale τ-module over $L(t)$ is equivalent to it being a "restricted module over $L(t)\{\tau\}$", and it further being pure of weight 0 is equivalent to it being "étale at $t^{-1} = 0$" by Lemma 4.7. (Note that in Definition 1.1 of [Tam95], the submodule $\mathcal{M}$ should also be an $O_{L(t)}$-sublattice of M). Tamagawa's analogue of the Tate conjecture [Tam95, Theorem 2.1] then says that (4.2) is an isomorphism of (finite dimensional) vector spaces over $k(\!(t^{-1})\!)$. Tamagawa theorem, whose proof is based on methods arising from p-adic Hodge theory, are only sketched in [Tam95]; details have since been provided by N. Stalder [Sta10].

Now consider the special case where the τ-module M' is $L(t)$ with $\tau_{M'} = \sigma|_{L(t)}$. So $\widehat{V}(M') = k(\!(t^{-1})\!)$ with the trivial Gal_L-action. We have isomorphisms

$$\mathrm{Hom}(M', M) = \mathrm{Hom}_{L(t)}(L(t), M)^{\tau} \xrightarrow{\sim} M^{\tau}, \quad f \mapsto f(1)$$

and

$$\mathrm{Hom}_{k(\!(t^{-1})\!)[\mathrm{Gal}_L]}\big(\widehat{V}(M'), \widehat{V}(M)\big) = \mathrm{Hom}_{k(\!(t^{-1})\!)[\mathrm{Gal}_L]}\big(k(\!(t^{-1})\!), \widehat{V}(M)\big) \xrightarrow{\sim} \widehat{V}(M)^{\mathrm{Gal}_L}, \quad f \mapsto f(1).$$

Combining with the isomorphism (4.2), we find that the natural map

$$M^{\tau} \otimes_{k(t)} k(\!(t^{-1})\!) \to \widehat{V}(M)^{\mathrm{Gal}_L} = \big((L^{\mathrm{sep}}(\!(t^{-1})\!) \otimes_{L(t)} M)^{\tau}\big)^{\mathrm{Gal}_L}$$

is an isomorphism of finite dimensional vector space over $k(\!(t^{-1})\!)$. $\square$

Corollary 4.9. *Let L be a finitely generated extension of k. Let M be an étale τ-module over $L(t)$ that is pure of some weight λ. Then for any separably closed extension K of L, the natural map*

$$\mathrm{End}(M) \otimes_{k(t)} k(\!(t^{-1})\!) \to \mathrm{End}(K(\!(t^{-1})\!) \otimes_{L(t)} M)^{\mathrm{Gal}(K/L)}$$

is an isomorphism.

Proof. Fix an embedding $L^{\mathrm{sep}} \subseteq K$. We have an inclusion $\mathrm{End}(L^{\mathrm{sep}}(\!(t^{-1})\!) \otimes_{L(t)} M) \subseteq \mathrm{End}(K(\!(t^{-1})\!) \otimes_{L(t)} M)$ of finite dimensional vector spaces over $k(\!(t^{-1})\!)$; it is actually an equality since by Proposition 4.5(ii), their dimensions depend only the weights of M. Hence,

$$\mathrm{End}(K(\!(t^{-1})\!) \otimes_{L(t)} M)^{\mathrm{Gal}(K/L)} = \mathrm{End}(L^{\mathrm{sep}}(\!(t^{-1})\!) \otimes_{L(t)} M)^{\mathrm{Gal}(K/L)} = \mathrm{End}(L^{\mathrm{sep}}(\!(t^{-1})\!) \otimes_{L(t)} M)^{\mathrm{Gal}_L}.$$

So without loss of generality, we may assume that $K = L^{\mathrm{sep}}$. Define the $L(t)$-module $H = \mathrm{End}_{L(t)}(M)$. Since M is an étale τ-module, we can give H the structure of étale τ-module over $L(t)$. The natural map $H^{\tau} \otimes_{k(t)} k(\!(t^{-1})\!) \to \big((L^{\mathrm{sep}}(\!(t^{-1})\!) \otimes_{L(t)} H)^{\tau}\big)^{\mathrm{Gal}_L}$ can be rewritten as

$$\mathrm{End}(M) \otimes_{k(t)} k(\!(t^{-1})\!) \to \mathrm{End}(L^{\mathrm{sep}}(\!(t^{-1})\!) \otimes_{L(t)} M)^{\mathrm{Gal}_L}.$$

So the corollary will follow from Theorem 4.8 if we can show that H is pure of weight 0.

The dual $M^{\vee} := \mathrm{Hom}_{L(t)}(M, L(t))$ is an étale τ-module over $L(t)$ that is pure of weight $-\lambda$ (for the weight, one can use the characterization in terms of eigenvalues as in Proposition 4.5). If M_1 and M_2 are étale τ-modules over $L(t)$ that are pure of weight λ_1 and λ_2, respectively, then $M_1 \otimes_{L(t)} M_2$ is pure of weight $\lambda_1 + \lambda_2$ (use Lemma 4.6). Therefore H, which is isomorphic as a τ-module to $M^{\vee} \otimes_{L(t)} M$, is pure of weight $-\lambda + \lambda = 0$. $\square$

4.4. **Proof of Theorem 4.1.** Let $\phi\colon A \to L[t]$ be a Drinfeld module with generic characteristic and L a finitely generated field.

Case 1: *Suppose that $A = k[t]$ and $F = k(t)$.*

Define $M_\phi := L[\tau]$ and give it the $L[t] = L \otimes_k A$-module structure for which

$$(c \otimes a) \cdot m = cm\phi_a$$

for $c \in L$, $a \in A$ and $m \in M_\phi$. Define the map $\tau_{M_\phi}\colon M_\phi \to M_\phi$ by $m \mapsto \tau m$. The pair (M_ϕ, τ_{M_ϕ}) is a τ-module over $L[t]$. As an $L[t]$-module, M_ϕ is free of rank n with basis $\beta = \{1, \tau, \ldots, \tau^{n-1}\}$. With respect to the basis β, the linearization $\tau_{M_\phi,\mathrm{lin}}$ is described by the $n \times n$ matrix

$$B := \begin{pmatrix} 0 & & 0 & (t-b_0)/b_n \\ 1 & & 0 & -b_1/b_n \\ & \ddots & & \vdots \\ 0 & & 1 & -b_{n-1}/b_n \end{pmatrix}$$

where $\phi_t = \sum_{i=0}^n b_i\tau^i$.

For $f \in \mathrm{End}_L(\phi)$, the map $M_\phi \to M_\phi$, $m \mapsto mf$ is a homomorphism of $L[t]$-modules which commutes with τ_{M_ϕ}. This gives a homomorphism $\mathrm{End}_L(\phi)^{\mathrm{opp}} \to \mathrm{End}(M_\phi)$ of $k[t]$-algebras; it is in fact an isomorphism [And86, Theorem 1]. Note that for a ring R, we will denote by R^{opp} the ring R with the same addition and multiplication $\alpha \cdot \beta = \beta\alpha$

Let $M_\phi(t)$ be the τ-module obtained by base extending M_ϕ to $L(t)$. Since $M_\phi(t)$ is an $L(t)$-vector space of dimension n with $\det(B) \in L(t)^\times$, we find that $M_\phi(t)$ is an étale τ-module. We have an isomorphism $\mathrm{End}_L(\phi)^{\mathrm{opp}} \otimes_{k[t]} k(t) = \mathrm{End}(M_\phi(t))$ of $k(t)$-algebras. From Anderson [And86, Prop. 4.1.1], we know that $M_\phi(t)$ is pure of weight $1/n$ (use Lemma 4.6 to relate his notion of purity and weight with ours).

Define $\overline{M}_\phi := \overline{L}(\!(\tau^{-1})\!)$. For $c = \sum_i a_i t^{-i} \in \overline{L}(\!(t^{-1})\!)$ and $m \in \overline{M}_\phi$, we define

$$c \cdot m = \sum_i a_i m \phi_t^{-i};$$

this turns $\overline{M}_\phi$ into a free $\overline{L}(\!(t^{-1})\!)$-module with basis $\{1, \tau, \ldots, \tau^{n-1}\}$. The pair $(\overline{M}_\phi, \tau_{\overline{M}_\phi})$, where $\tau_{\overline{M}_\phi}\colon \overline{M}_\phi \to \overline{M}_\phi$ is the map $m \mapsto \tau m$, is a τ-module. One readily verifies that $\overline{M}_\phi$ agrees with the base extension of M_ϕ to $\overline{L}(\!(t^{-1})\!)$.

Take any $f \in D_\phi$. Since f commutes with ϕ_t, we find that the map $\overline{M}_\phi \to \overline{M}_\phi$, $m \mapsto mf$ is a homomorphism of $\overline{L}(\!(t^{-1})\!)$-modules which commutes with $\tau_{\overline{M}_\phi}$. This gives a homomorphism

$$D_\phi^{\mathrm{opp}} \hookrightarrow \mathrm{End}(\overline{M}_\phi) \tag{4.3}$$

of F_∞-algebras. By Lemma 2.2(ii) and Proposition 4.5, D_ϕ^{opp} and $\mathrm{End}(\overline{M}_\phi)$ are both F_∞-division algebras with invariant $1/n$, so (4.3) is an isomorphism. Moreover, the isomorphism (4.3) is compatible with the respective $\mathrm{Gal}(\overline{L}/L)$-actions. Restricting (4.3) to $\mathrm{End}_L(\phi) \otimes_{k[t]} k(\!(t^{-1})\!)$ gives the isomorphism

$$\mathrm{End}_L(\phi)^{\mathrm{opp}} \otimes_{k[t]} k(\!(t^{-1})\!) \xrightarrow{\sim} \mathrm{End}(M_\phi) \otimes_{k[t]} k(\!(t^{-1})\!) = \mathrm{End}(M_\phi(t)) \otimes_{k(t)} k(\!(t^{-1})\!).$$

By Lemma 4.2, it suffices to prove that $D_\phi^{\mathrm{Gal}(\overline{L}/L)} = \mathrm{End}_L(\phi) \otimes_{k[t]} k(\!(t^{-1})\!)$, which we find is equivalent to showing that the natural map

$$\mathrm{End}(M_\phi(t)) \otimes_{k(t)} k(\!(t^{-1})\!) \to \mathrm{End}(\overline{M}_\phi)^{\mathrm{Gal}(\overline{L}/L)} = \mathrm{End}(\overline{L}(\!(t^{-1})\!) \otimes_{L(t)} M_\phi(t))^{\mathrm{Gal}(\overline{L}/L)}$$

is an isomorphism. Since $M_\phi(t)$ is an étale τ-module that is pure of weight $1/n$ and L is finitely generated, this follows from Corollary 4.9.

Case 2: *General case.*

Choose a non-constant element $t \in A$. Composing the inclusion $k[t] \subseteq A$ with ϕ gives a ring homomorphism
$$\phi'\colon k[t] \to L[\tau],\ a \mapsto \phi'_a.$$
By (1.1), we have $\operatorname{ord}_{\tau^{-1}}(\phi'_t) < 0$ and hence ϕ' is a Drinfeld module (though possibly of a different rank than ϕ). Since ϕ has generic characteristic, so does ϕ'. Let ∞ also denote the place of $k(t)$ with uniformizer t^{-1}.

Since $\phi(A) \supseteq \phi'(k[t])$, we have inclusions
$$\operatorname{End}_L(\phi) \subseteq \operatorname{End}_L(\phi') \quad \text{and} \quad D_\phi \subseteq D_{\phi'}.$$
Therefore,
$$\operatorname{End}_L(\phi) \otimes_A F_\infty \subseteq D_\phi^{\operatorname{Gal}(\bar{L}/L)} \subseteq D_{\phi'}^{\operatorname{Gal}(\bar{L}/L)} = \operatorname{End}_L(\phi') \otimes_A F_\infty$$
where the equality follows from Case 1. By Lemma 4.2, it thus suffices to prove the inclusion $\operatorname{End}_L(\phi) \supseteq \operatorname{End}_L(\phi')$. The ring $\operatorname{End}_L(\phi')$ certainly contains $\phi(A)$. Since ϕ' has generic characteristic, the ring $\operatorname{End}_L(\phi')$ is commutative [Dri74, §2]. So $\operatorname{End}_L(\phi')$ is a subring of $L[\tau]$ that commutes with $\phi(A)$; it is thus a subset of $\operatorname{End}_L(\phi)$.

5. Proof of Theorem 1.1

Let $\phi\colon A \to L[\tau]$ be a Drinfeld module of generic characteristic and rank n. Assume that L is a finitely generated field and that $\operatorname{End}_{\bar{L}}(\phi) = \phi(A)$. To ease notation, we set $D := D_\phi$ which is a central F_∞-division algebra with invariant $-1/n$. Several times in the proof, we will replace L by a finite extension; this is fine since we are only interested in $\rho_\infty(W_L)$ up to commensurability. The $n = 1$ case has already been proved (Corollary 3.2), so we may assume that $n \geq 2$.

5.1. **Zariski denseness.** Let GL_D be the algebraic group defined over F_∞ such that $\operatorname{GL}_D(R) = (D \otimes_{F_\infty} R)^\times$ for a commutative F_∞-algebra R. In particular, we have $\rho_\infty(W_L) \subseteq D^\times = \operatorname{GL}_D(F_\infty)$. The main task of this section is to prove the following.

Proposition 5.1. *With assumptions as above, $\rho_\infty(W_L)$ is Zariski dense in GL_D.*

Let $\mathbb{G}$ be the algebraic subgroup of GL_D obtained by taking the Zariski closure of $\rho_\infty(W_L)$ in GL_D; it is defined over F_∞. After replacing L by a finite extension, we may assume that $\mathbb{G}$ is connected. Choose an algebraically closed extension K of F_∞. For an algebraic group G over F_∞, we will denote by G_K the algebraic group over K obtained by base extension. We need to prove that $\mathbb{G} = \operatorname{GL}_D$, or equivalently that $\mathbb{G}_K = \operatorname{GL}_{D,K}$.

An isomorphism $D \otimes_{F_\infty} K \cong M_n(K)$ of K-algebras induces an isomorphism $\operatorname{GL}_{D,K} \cong \operatorname{GL}_{n,K}$ of algebraic groups over K (both are unique up to an inner automorphism). We fix such an isomorphism, which we use as an identification $\operatorname{GL}_{D,K} = \operatorname{GL}_{n,K}$ and this gives us an action of $D \otimes_{F_\infty} K$ on K^n.

We will use the following criterion of Pink to show that $\mathbb{G}_K$ and $\operatorname{GL}_{D,K} = \operatorname{GL}_{n,K}$ are equal.

Lemma 5.2 ([Pin97, Proposition A.3])**.** *Let K be an algebraically closed field and let $G \subseteq \operatorname{GL}_{n,K}$ be a reductive connected linear algebraic group acting irreducibly on K^n. Suppose that G has a cocharacter which has weight 1 with multiplicity 1 and weight 0 with multiplicity $n-1$ on K^n. Then $G = \operatorname{GL}_{n,K}$.*

Lemma 5.3. *With our fixed isomorphism, the algebraic group $\mathbb{G}_K$ acts irreducibly on K^n.*

Proof. Let B be the F_∞-vector subspace of D generated by $\rho_\infty(W_L)$. Using that $\rho_\infty(W_L)$ is a group and that every element of D is algebraic over F_∞, we find that B is a division algebra whose center contains F_∞. By our analogue of the Tate conjecture (Theorem 4.1) and our assumption $\mathrm{End}_{\overline{L}}(\phi) = \phi(A)$, we have

$$\mathrm{Cent}_D(B) = \mathrm{Cent}_D(\rho_\infty(W_L)) = F_\infty.$$

By the Double Centralizer Theorem, we have $B = \mathrm{Cent}_D(\mathrm{Cent}_D(B))$ and hence $B = \mathrm{Cent}_D(F_\infty) = D$.

Let H be a non-zero K-subspace of K^n that is stable under the action of $\mathbb{G}_K$. Since $\rho_\infty(W_L) \subseteq \mathbb{G}(F_\infty)$ and $F_\infty \subseteq K$, we find that H is stable under the action of $B \otimes_{F_\infty} K = D \otimes_{F_\infty} K \cong M_n(K)$. Therefore, $H = K^n$. □

By Lemma 5.3 and the following lemma, we deduce that $\mathbb{G}_K$ is reductive.

Lemma 5.4 ([Pin97, Fact A.1]). *Let K be an algebraically closed field, and let $G \subseteq \mathrm{GL}_{n,K}$ be a connected linear algebraic group. If G acts irreducibly on the vector space K^n, then G is reductive.*

Let X be a model of L as described in §1.2. For a fixed closed point x of X, choose a matrix $h_x \in \mathrm{GL}_n(F)$ with characteristic polynomial $P_{\phi,x}(T)$. Let $H_x \subseteq \mathrm{GL}_{n,F}$ denote the Zariski closure of the group generated by h_x, and let T_x be the identity component of H_x. Since F has positive characteristic, some positive power of h_x will be semisimple. The algebraic group T_x is thus an algebraic torus which is called the Frobenius torus at x. The following result of Pink describes what happens when ϕ has ordinary reduction at x.

Recall that by reducing the coefficients of ϕ, we obtain a Drinfeld module $\phi_x\colon A \to \mathbb{F}_x[\tau]$ of rank n. Let λ_x be the place of F corresponding to the characteristic of ϕ_x. The Tate module $T_{\lambda_x}(\phi_x)$ is a free $\mathcal{O}_{\lambda_x}$-module of rank n_x, where n_x is an integer *strictly* less than n. We say that ϕ has ordinary reduction at x if $n_x = n-1$.

Lemma 5.5 ([Pin97, Lemma 2.5]). *If ϕ has ordinary reduction at $x \in X$, then $T_x \subseteq \mathrm{GL}_{n,F}$ possesses a cocharacter over $\overline{F}$ which in the given representation has weight 1 with multiplicity 1, and weight 0 with multiplicity $n-1$.*

Lemma 5.6 ([Pin97, Corollary 2.3]). *The set of closed points of X for which ϕ has ordinary reduction has positive Dirichlet density.*

We can now finish the proof of Proposition 5.1. We have shown that $\mathbb{G}_K$ is a reductive, connected, linear algebraic group acting irreducibly on K^n. By Lemma 5.2 it suffices to show that $\mathbb{G}_K$ has a cocharacter which has weight 1 with multiplicity 1 and weight 0 with multiplicity $n-1$ on K^n.

By Lemma 5.6, there exists a closed point x of X for which ϕ has ordinary reduction. Some common power of h_x and $\rho_\infty(\mathrm{Frob}_x)$ are conjugate in $\mathrm{GL}_n(K)$ because they will be semisimple with the same characteristic polynomial. So with our fixed isomorphism $\mathrm{GL}_{D,K} = \mathrm{GL}_{n,K}$, we find that $T_{x,K}$ is conjugate to an algebraic subgroup of $\mathbb{G}_K$. The desired cocharacter of $\mathbb{G}_K$ is then obtained by appropriately conjugating the cocharacter coming from Lemma 5.5.

5.2. **Open commutator subgroup.** Let SL_D be the kernel of the homomorphism $\mathrm{GL}_D \to \mathbb{G}_{m,F_\infty}$ arising from the reduced norm. Let PGL_D and PSL_D be the algebraic groups obtained by quotienting GL_D and SL_D, respectively, by their centers. As linear algebraic groups, SL_D is simply connected and PGL_D is adjoint. The natural map $\mathrm{PSL}_D \to \mathrm{PGL}_D$ is an isomorphism of algebraic groups and hence the homomorphism $\mathrm{SL}_D \to \mathrm{PGL}_D$ is a universal cover.

The commutator morphism of GL_D factors through a unique morphism

$$[\ ,\ ]\colon \mathrm{PGL}_D \times \mathrm{PGL}_D \to \mathrm{SL}_D\,.$$

Let Γ be the closure of the image of $\rho_\infty(W_L)$ in $\mathrm{PGL}_D(F_\infty)$. Let $\Gamma' \subseteq \mathrm{SL}_D(F_\infty)$ be the closure of the subgroup generated by $[\Gamma, \Gamma]$. (Both closures are with respect to the ∞-adic topology.)

The group Γ is compact since it is closed and $\operatorname{PGL}_D(F_\infty)$ is compact. The group Γ is Zariski dense in PGL_D by Proposition 5.1. If we were working over a local field of characteristic 0, this would be enough to deduce that Γ is an open subgroup of $\operatorname{PGL}_D(F_\infty)$. However, in the positive characteristic setting the Lie theory is more complicated. Fortunately, what we need has already been worked out by Pink.

Theorem 0.2(c) of [Pin98] says that there is a closed subfield E of F_∞, an absolutely simple adjoint linear group H over E, and an isogeny $f\colon H \times_E F_\infty \to \operatorname{PGL}_D$ with nowhere vanishing derivative such that Γ' is the image under $\widetilde{f}$ of an open subgroup of $\widetilde{H}(E)$ where $\widetilde{f}\colon \widetilde{H} \times_E F_\infty \to \operatorname{SL}_D$ is the associated isogeny of universal covers.

The following lemma will be needed to show that $E = F_\infty$. Let

$$\operatorname{Ad}_{\operatorname{PGL}_D}\colon \operatorname{PGL}_D \to \operatorname{GL}_{m,F_\infty}$$

be the adjoint representation of PGL_D where m is the dimension of PGL_D.

Lemma 5.7. *Let $\mathcal{O} \subseteq F_\infty$ be the closure of the subring generated by 1 and $\operatorname{tr}(\operatorname{Ad}_{\operatorname{PGL}_D}(\Gamma))$. Then the quotient field of $\mathcal{O}$ is F_∞.*

Proof. We will consider $\operatorname{Ad}_{\operatorname{PGL}_D}$ at Frobenius elements, and thus reduce to a result of Pink. Take any element $\alpha \in D^\times$. Let $\alpha_1, \ldots, \alpha_n \in \overline{F}_\infty$ be the roots of the (reduced) characteristic polynomial $\det(TI - \alpha)$. We have

$$\operatorname{tr}(\operatorname{Ad}_{\operatorname{PGL}_D}(\alpha)) = \Big(\sum_{i=1}^{n} \alpha_i\Big)\Big(\sum_{j=1}^{n} {\alpha_j}^{-1}\Big) - 1 = \operatorname{tr}(\alpha)\cdot \operatorname{tr}(\alpha^{-1}) - 1$$

(one need only check the analogous result for $\operatorname{PGL}_{n,\overline{F}_\infty}$ since it is isomorphic to $\operatorname{PGL}_{D,\overline{F}_\infty}$). For each closed point x of X, define $a_x := \operatorname{tr}(\operatorname{Ad}_{\operatorname{PGL}_D}(\rho_\infty(\operatorname{Frob}_x)))$. We have $a_x = \operatorname{tr}(\rho_\lambda(\operatorname{Frob}_x))\cdot \operatorname{tr}(\rho_\lambda(\operatorname{Frob}_x)^{-1}) - 1$ for any place $\lambda \neq \lambda_x$ of F, and hence a_x belongs to F. By [Pin97, Proposition 2.4], the field F is generated by the set $\{a_x\}_x$ where x varies over the closed points of X (this requires our assumptions that $\operatorname{End}_{\overline{L}}(\phi) = \phi(A)$ and $n \geq 2$). Therefore, the quotient field of $\mathcal{O}$ is F_∞. ☐

Lemma 5.7 along with [Pin98, Proposition 0.6(c)] shows that $E = F_\infty$ and that $f\colon H \to \operatorname{PGL}_D$ is an isomorphism. Therefore, Γ' is an open subgroup of $\operatorname{SL}_D(F_\infty)$.

Proposition 5.8. *The group $\rho_\infty(W_L)$ contains an open subgroup of $\operatorname{SL}_D(F_\infty)$.*

Proof. The group $\rho_\infty(W_{L\bar{k}})$ is a normal subgroup of $\rho_\infty(W_L)$ with abelian quotient; it is also compact since $W_{L\bar{k}} = \operatorname{Gal}(L^{\mathrm{sep}}/L\bar{k})$ is compact and ρ_∞ is continuous. Therefore, Γ' is a subgroup of $\rho_\infty(W_{L\bar{k}})$. The proposition follows since we just showed that Γ' is open in $\operatorname{SL}_D(F_\infty)$. ☐

5.3. **End of the proof.** We have $\rho_\infty(W_L) \subseteq D^\times$. In Proposition 5.8, we showed that $\rho_\infty(W_L)$ contains an open subgroup of $\operatorname{SL}_D(F_\infty) = \{\alpha \in D^\times : \det(\alpha) = 1\}$. To complete the proof of Theorem 1.1, it suffices to show that $\det(\rho_\infty(W_L))$ is an open subgroup with finite index in $F_\infty^\times$.

Lemma 5.9. *The image of the the reduced norm map $\det \circ \rho_\infty\colon W_L \to F_\infty^\times$ is an open subgroup of finite index in $F_\infty^\times$*

Proof. One can construct a "determinant" Drinfeld module of ϕ; it is a rank 1 Drinfeld module $\psi\colon A \to L[\tau]$ and has the property that $\bigwedge_{\mathcal{O}_\lambda}^n T_\lambda(\phi)$ and $T_\lambda(\psi)$ are isomorphic $\mathcal{O}_\lambda[\operatorname{Gal}_L]$-modules for every place $\lambda \neq \infty$ of F. This can accomplished by following G. Anderson and working in the larger category of A-motives where one can take tensor products. A proof of the existence of such a ψ can be found in [vdH04, Theorem 3.3] and the isomorphism of Tate modules is then straightforward.

Let X be a model of L as described in §1.2. For each closed point x of X and place $\lambda \neq \lambda_x, \infty$ of F, we thus have

$$\det(\rho_{\phi,\infty}(\mathrm{Frob}_x)) = \det(\rho_{\phi,\lambda}(\mathrm{Frob}_x)) = \rho_{\psi,\lambda}(\mathrm{Frob}_x) = \rho_{\psi,\infty}(\mathrm{Frob}_x).$$

By the Chebotarev density theorem, that $\det \circ \rho_{\phi,\infty}(\mathrm{Frob}_x)$ equals $\rho_{\psi,\infty}(\mathrm{Frob}_x)$ for all closed points x of X implies that $\det \circ \rho_{\phi,\infty}$ equals $\rho_{\psi,\infty}$. The lemma now follows from Corollary 3.2 since ψ has rank 1. $\square$

6. Proof of Theorem 1.2

By [Dri74, p.569 Corollary] and our generic characteristic assumption, the ring $A' := \mathrm{End}_{\overline{L}}(\phi)$ is a projective A-module and $F'_\infty := A' \otimes_A F_\infty$ is a field satisfying $[F'_\infty : F_\infty] \leq n$. Let F' be the quotient field of A'. There is a unique place of F' lying over the place ∞ of F, which we shall also denote it by ∞, and F'_∞ is indeed the completion of F' at ∞.

After replacing L by a finite extension, we may assume that A' equals $\mathrm{End}_L(\phi)$. Identifying A with its image in $L[\tau]$, the inclusion map

$$\phi' \colon A' \to L[\tau].$$

extends ϕ. The homomorphism ϕ' need not be a Drinfeld module, at least according to our definition, since A' need not be a maximal order in F'. Instead of extending our definition of Drinfeld module, we follow Pink and Hayes, and adjust ϕ' by an appropriate isogeny.

Let B be the normalization of A' in F'; it is a maximal order of F' consisting of functions that are regular away from ∞. By [Hay79, Prop. 3.2], there is a Drinfeld module $\psi \colon B \to \overline{L}[\tau]$ and a non-zero $f \in L[\tau]$ such that $f\phi'(x) = \psi(x)f$ for all $x \in A'$. The Drinfeld module ψ has rank $n' = n/[F' : F]$ and $\mathrm{End}_{\overline{L}}(\psi) = \psi(B)$. After replacing L by a finite extension, we may assume that $\psi(B) \subseteq L[\tau]$.

It is straightforward to show that the map $\mathrm{Cent}_{\overline{L}(\!(\tau^{-1})\!)}(\psi(A')) \to \mathrm{Cent}_{\overline{L}(\!(\tau^{-1})\!)}(\mathrm{End}_{\overline{L}}(\phi))$ defined by $v \mapsto f^{-1}vf$ is a bijection, and hence we have an isomorphism

$$D_\psi \xrightarrow{\sim} \mathrm{Cent}_{D_\phi}(\mathrm{End}_{\overline{L}}(\phi)) =: B_\phi, \quad v \mapsto f^{-1}vf.$$

The corresponding representations ρ_∞ are compatible under this map.

Lemma 6.1. *For all $\sigma \in W_L$, we have $\rho_{\psi,\infty}(\sigma) = f^{-1}\rho_{\phi,\infty}(\sigma)f$.*

Proof. Choose any $u \in \overline{L}(\!(\tau^{-1})\!)^\times$ such that $u^{-1}\psi(F'_\infty)u \subseteq \bar{k}(\!(\tau^{-1})\!)$. So $u^{-1}f\phi(A)f^{-1}u \subseteq \bar{k}(\!(\tau^{-1})\!)$ and hence $(f^{-1}u)^{-1}\phi(F_\infty)(f^{-1}u) \subseteq \bar{k}(\!(\tau^{-1})\!)$. Therefore,

$$\rho_{\psi,\infty}(\sigma) = \sigma(f^{-1}u)\tau^{\deg(\sigma)}(f^{-1}u)^{-1} = f^{-1}\sigma(u)\tau^{\deg(\sigma)}u^{-1}f = f^{-1}\rho_{\phi,\infty}(\sigma)f. \qquad \square$$

Therefore, $\rho_{\phi,\infty}(W_L)$ is an open subgroup of finite index in $\mathrm{Cent}_{D_\phi}(\mathrm{End}_{\overline{L}}(\phi))^\times$ if and only if $\rho_{\psi,\infty}(W_L)$ is an open subgroup of finite index in $D_\psi^\times$. However $\mathrm{End}_{\overline{L}}(\psi) = \psi(B)$, so $\rho_{\psi,\infty}(W_L)$ is an open subgroup of finite index in $D_\psi^\times$ by Theorem 1.1 which we proved in §5.

7. Proof of Theorem 1.5

To ease notation, set $D = D_\phi$ and define the (surjective) valuation $v \colon D \to \mathbb{Z} \cup \{+\infty\}$, $\alpha \mapsto \mathrm{ord}_{\tau^{-1}}(\alpha)/d_\infty$. Let $\mathcal{O}_D$ be the valuation ring of D with respect to v and let $\mathfrak{P}$ denote it maximal ideal. We have fixed a uniformizer π of F_∞ that we can view as element of D by identifying it with ϕ_π. Let $\mu_{D^\times}$ be a Haar measure for $D^\times$. We fix an open subset U of $\mathcal{O}_\infty$, and let $\mathcal{C}$ be the set of $\alpha \in D^\times$ for which $\mathrm{tr}(\alpha) \in U$. We also fix an integer $0 \leq i < n$, and let $\mathcal{V}_i$ be the set of $\alpha \in D^\times$ that satisfy $v(\alpha) = -i$.

Take any positive integer $d \equiv i \pmod{n}$ that is divisble by $[\mathbb{F}_L : \mathbb{F}_\infty]$. Let x be a closed point of X of degree d. We have $v(\rho_\infty(\mathrm{Frob}_x)) = -\deg(\mathrm{Frob}_x)/d_\infty = -[\mathbb{F}_x : \mathbb{F}_\infty] = -d$. Therefore,

$$v(\rho_\infty(\mathrm{Frob}_x)\pi^{\lfloor d/n \rfloor}) = -d + \lfloor d/n \rfloor \operatorname{ord}_{\tau^{-1}}(\phi_\pi)/d_\infty = -d + \lfloor d/n \rfloor n = -i.$$

So $\rho_\infty(\mathrm{Frob}_x)\pi^{\lfloor d/n \rfloor}$ belongs to $\mathcal{V}_i$; it belongs to $\mathcal{C}$ if and only if $a_x(\phi)\pi^{\lfloor d/n \rfloor}$ is in U.

Let

$$\overline{\rho}\colon\ \mathrm{Gal}_L \to D^\times/\langle \pi \rangle$$

be the continuous homomorphism obtained by composing $\rho_\infty\colon W_L \to D^\times$ with the quotient map to $D^\times/\langle \pi \rangle$, and then using the compactness of $D^\times/\langle \pi \rangle$ to extend by continuity.

We can identify $\mathcal{V}_i$, and hence also identify $\mathcal{V}_i \cap \mathcal{C}$, with its image in $D^\times/\langle \pi \rangle$. This shows that

$$\{x \in |X|_d : a_x(\phi)\pi^{\lfloor d/n \rfloor} \in U\} = \{x \in |X|_d : \overline{\rho}(\mathrm{Frob}_x) \subseteq \mathcal{V}_i \cap \mathcal{C}\},$$

which we can now estimate with the Chebotarev density theorem. By assumption, we have $\rho_\infty(W_L) = D^\times$ and hence $\rho_\infty(W_{L\bar{k}}) = \mathcal{O}_D^\times$ by Lemma 2.2(iv). Therefore, $\overline{\rho}(\mathrm{Gal}_L) = D^\times/\langle \pi \rangle$ and the cosets of $\overline{\rho}(\mathrm{Gal}_{L\bar{k}})$ in $D^\times/\langle \pi \rangle$ are the sets $\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_{n-1}$. By the global function field version of the Chebotarev density theorem, we have

$$\lim_{\substack{d \equiv i \pmod{n},\, d \equiv 0 \pmod{[\mathbb{F}_L:\mathbb{F}_\infty]} \\ d \to +\infty}} \frac{|\{x \in |X|_d : \overline{\rho}(\mathrm{Frob}_x) \subseteq \mathcal{V}_i \cap \mathcal{C}\}|}{\#|X|_d} = \frac{\mu_{D^\times}(\mathcal{V}_i \cap \mathcal{C})}{\mu_{D^\times}(\mathcal{V}_i)}.$$

It remains to compute the value $\mu_{D^\times}(\mathcal{V}_i \cap \mathcal{C})/\mu_{D^\times}(\mathcal{V}_i)$.

We first need to recall some facts about the division algebra D, cf. [Rie70, §2] for some background and references. The algebra D contains an unramified extension W of F_∞ of degree n and an element β such that

$$D = W \oplus W\beta \oplus \cdots \oplus W\beta^{n-1}$$

where β^n is a uniformizer of F_∞ and the map $a \mapsto \beta a \beta^{-1}$ generates $\mathrm{Gal}(W/F_\infty)$. Define the map

$$f\colon W^n \xrightarrow{\sim} D, \quad (a_0, \ldots, a_{n-1}) \mapsto \sum\nolimits_{i=0}^{n-1} a_i \beta^i;$$

it is an isomorphism of (left) vector spaces over W. Let $\mathcal{O}_W$ be the ring of integers of W and denote its maximal ideal by $\mathfrak{p}$. For any integers $m \in \mathbb{Z}$ and $0 \le j < n$, we have

$$\mathfrak{P}^{mn+j} = f((\mathfrak{p}^{m+1})^j \times (\mathfrak{p}^m)^{n-j}).$$

For $\alpha \in D$, the *reduced trace* $\operatorname{tr}(\alpha)$ is the trace of the endomorphism of the W-vector space D given by $v \mapsto v\alpha$. One can check that $\operatorname{tr}(f(a_0, \ldots, a_{n-1})) = \mathrm{Tr}_{W/F_\infty}(a_0)$ for $(a_0, \ldots, a_{n-1}) \in W^n$.

First consider the case $i \ge 1$. We have

$$\mathcal{V}_i = \mathfrak{P}^{-i} - \mathfrak{P}^{-(i-1)} = f(\mathcal{O}_W^{n-i} \times (\mathfrak{p}^{-1} - \mathcal{O}_W) \times (\mathfrak{p}^{-1})^{i-1})$$

and the measures arising from the restriction of the Haar measures of $D^\times$ and W^n, respectively, agree (up to a constant factor). So

$$\frac{\mu_{D^\times}(\mathcal{V}_i \cap \mathcal{C})}{\mu_{D^\times}(\mathcal{V}_i)} = \mu_{\mathcal{O}_W}(\{a_0 \in \mathcal{O}_W : \mathrm{Tr}_{W/F_\infty}(a_0) \in U\})$$

where $\mu_{\mathcal{O}_W}$ is the Haar measure normalized so that $\mu_{\mathcal{O}_W}(\mathcal{O}_W) = 1$. Since $\mathrm{Tr}_{W/F_\infty}\colon \mathcal{O}_W \to \mathcal{O}_\infty$ is a surjective homomorphism of $\mathcal{O}_\infty$-modules, we have $\mu_{\mathcal{O}_W}(\{a_0 \in \mathcal{O}_W : \mathrm{Tr}_{W/F_\infty}(a_0) \in U\}) = \mu(U)$.

Now consider the case $i = 0$. We have

$$\mathcal{V}_0 = \mathcal{O}_D - \mathfrak{P} = f((\mathcal{O}_W - \mathfrak{p}) \times \mathcal{O}_W^{n-1}).$$

and the measures arising from the restriction of the Haar measures of $D^\times$ and W^n, respectively, agree (up to a constant factor). So

$$\frac{\mu_{D^\times}(\mathcal{V}_0 \cap \mathcal{C})}{\mu_{D^\times}(\mathcal{V}_0)} = \frac{\mu_{\mathcal{O}_W}(\{a_0 \in \mathcal{O}_W - \mathfrak{p} : \mathrm{Tr}_{W/F_\infty}(a_0) \in U\})}{\mu_{\mathcal{O}_W}(\mathcal{O}_W - \mathfrak{p})}.$$

Note that $\mathrm{Tr}_{W/F_\infty} : \mathcal{O}_W \to \mathcal{O}_\infty$ is a surjective homomorphism of $\mathcal{O}_\infty$-modules satisfying $\mathrm{Tr}_{W/F_\infty}(\mathfrak{p}) = \pi\mathcal{O}_\infty$. Fix a coset κ of $\pi\mathcal{O}_\infty$ in $\mathcal{O}_\infty$. Then $\mathrm{Tr}^{-1}_{W/F_\infty}(\kappa) \cap (\mathcal{O}_W - \mathfrak{p})$ is the union of $q^{d_\infty(n-1)}$ cosets of $\mathfrak{p}$ in $\mathcal{O}_W$ when $\kappa \neq \pi\mathcal{O}_\infty$, and $q^{d_\infty(n-1)} - 1$ cosets when $\kappa = \pi\mathcal{O}_\infty$. One can then check that $\mu_{\mathcal{O}_W}(\{a_0 \in \mathcal{O}_W - \mathfrak{p} : \mathrm{Tr}_{W/F_\infty}(a_0) \in U\})/\mu_{\mathcal{O}_W}(\mathcal{O}_W - \mathfrak{p}) = \nu(U)$ by taking into account this weighting of cosets.

The following lemma will be used in the next section.

Lemma 7.1. *For $j \geq 1$, we have $\mu_{D^\times}(\{\alpha \in \mathcal{O}_D - \pi\mathcal{O}_D : \mathrm{tr}(\alpha) \equiv 0 \pmod{\pi^j\mathcal{O}_\infty}\}) \ll 1/q^{d_\infty j}$.*

Proof. We have $f(\mathcal{O}_W^n - \mathfrak{p}^n) = \mathcal{O}_D - \pi\mathcal{O}_D$. One can then show that

$$\begin{aligned}
&\mu_{D^\times}(\{\alpha \in \mathcal{O}_D - \pi\mathcal{O}_D : \mathrm{tr}(\alpha) \equiv 0 \pmod{\pi^j\mathcal{O}_\infty}\}) \\
\ll&\mu'(\{(a_0, \dots, a_{n-1}) \in \mathcal{O}_W^n - \mathfrak{p}^n : \mathrm{Tr}_{W/F_\infty}(a_0) \equiv 0 \pmod{\pi^j\mathcal{O}_\infty}\}) \\
\ll&\mu_{\mathcal{O}_W}(\{a_0 \in \mathcal{O}_W : \mathrm{Tr}_{W/F_\infty}(a_0) \equiv 0 \pmod{\pi^j\mathcal{O}_\infty}\})
\end{aligned}$$

where μ' is a fixed Haar measure of W^n. This last quantity is bounded by $|\mathcal{O}_\infty/\pi^j\mathcal{O}_\infty|^{-1} = q^{-d_\infty j}$. □

8. Proof of Theorem 1.7

To ease notation, set $D = D_\phi$ and define the (surjective) valuation $v\colon D \to \mathbb{Z} \cup \{+\infty\}$, $\alpha \mapsto \mathrm{ord}_{\tau^{-1}}(\alpha)/d_\infty$. Let $\mathcal{O}_D$ be the valuation ring of D with respect to v. Fix a uniformizer π of F_∞ that we can view as element of D by identifying it with ϕ_π.

For each $\alpha \in D^\times$, we define $e(\alpha)$ to be the smallest integer such that $\alpha\pi^{e(\alpha)}$ belongs to $\mathcal{O}_D$ (equivalently, $v(\alpha\pi^{e(\alpha)}) \geq 0$). Define the map

$$f\colon D^\times \to \mathcal{O}_\infty, \quad \alpha \mapsto \mathrm{tr}(\alpha\pi^{e(\alpha)})$$

where tr is the reduced trace. For each integer $j \geq 1$, let $f_j\colon D^\times \to \mathcal{O}_\infty/\pi^j\mathcal{O}_\infty$ be the function obtained by composing f with the reduction modulo π^j homomorphism.

Lemma 8.1. *Let x be a closed point of X of degree d. Then $f_j(\rho_\infty(\mathrm{Frob}_x)) = 0$ for all integers $1 \leq j \leq \mathrm{ord}_\infty(a_x(\phi)) + \lceil d/n \rceil$. In particular,*

$$P_{\phi,a}(d) \leq |\{x \in |X|_d : f_j(\rho_\infty(\mathrm{Frob}_x)) = 0\}|.$$

Proof. Set $\alpha := \rho_\infty(\mathrm{Frob}_x)$. We have $v(\alpha) = -\deg(x)/d_\infty = -d$. Since $v(\pi) = \mathrm{ord}_{\tau^{-1}}(\phi_\pi)/d_\infty = n$, we have $e(\alpha) = \lceil d/n \rceil$. Hence $f(\alpha) = \mathrm{tr}(\alpha\pi^{e(\alpha)}) = \mathrm{tr}(\alpha)\pi^{e(\alpha)} = a_x(\phi)\pi^{e(\alpha)}$, which is divisible by π^j for any integer $1 \leq j \leq \mathrm{ord}_\infty(a_x(\phi)) + e(\alpha)$. □

For each integer $j \geq 1$, define the group

$$G_j := D^\times/(F_\infty^\times(1 + \pi^j\mathcal{O}_D)).$$

If $\alpha, \beta \in D^\times$ are in the same coset of G_j, then $f_j(\alpha) = 0$ if and only if $f_j(\beta) = 0$ [observe that $f(\alpha\pi^i) = f(\alpha)$ for $i \in \mathbb{Z}$, $f(u\alpha) = uf(\alpha)$ for $u \in \mathcal{O}_\infty^\times$, and $f_j(\alpha(1 + \pi^j\gamma)) = f_j(\alpha)$ for $\gamma \in \mathcal{O}_D$]. So by abuse of notation, it makes sense to ask whether $f_j(\alpha) = 0$ for a coset $\alpha \in G_j$. The subset $C_j := \{\alpha \in G_j : f_j(\alpha) = 0\}$ of G_j is stable under conjugation. The group G_j and the set C_j do not depend on the initial choice of uniformizer π.

Let $\overline{\rho}_j\colon \operatorname{Gal}_L \to G_j$ be the Galois representation obtained by composing ρ_∞ with the quotient map to G_j and then extending to a representation of Gal_L by using that ρ_∞ is continuous and G_j is finite. Lemma 8.1 gives the bound

$$P_{\phi,a}(d) \le |\{x \in |X|_d : \overline{\rho}_j(\operatorname{Frob}_x) \subseteq C_j\}| \tag{8.1}$$

whenever $1 \le j \le \operatorname{ord}_\infty(a) + \lceil d/n \rceil$.

We shall bound $P_{\phi,a}(d)$ by bounding the right-hand side of (8.1) with an effective version of the Chebotarev density theorem and then choosing j to optimize the resulting bound. Let $\widetilde{G}_j$ be the image of $\overline{\rho}_j\colon \operatorname{Gal}_L \to G_j$ and let $\widetilde{C}_j$ be the intersection of $\widetilde{G}_j$ with C_j. The effective Chebotarev density theorem of Murty and Scherk [MS94, Théorème 2] implies that

$$|\{x \in |X|_d : \overline{\rho}_j(\operatorname{Frob}_x) \subseteq \widetilde{C}_j\}| \ll m_j \frac{|\widetilde{C}_j|}{|\widetilde{G}_j|} \cdot \#|X|_d + |\widetilde{C}_j|^{1/2}(1 + (\varrho_j + 1)|\mathscr{D}|)\frac{q^{d_\infty d/2}}{d}$$

where the implicit constant depends only on L, and the quantities m_j, $|\mathscr{D}|$ and ϱ_j will be described below. (Their theorem is only given for a conjugacy class, not a subset stable under conjugation, but one can easily extend to this case by using the techniques of [MMS88].)

We first bound the cardinality of our subset C_j.

Lemma 8.2. *We have $|C_j| \ll q^{d_\infty(n^2-2)j}$ and $|C_j|/|G_j| \ll 1/q^{d_\infty j}$.*

Proof. We first prove the bound of $|C_j|/|G_j|$. For $\alpha \in D^\times$, we have $\alpha\pi^{e(\alpha)} \in \mathcal{O}_D - \pi\mathcal{O}_D$ and hence

$$\frac{|C_j|}{|G_j|} \ll \mu_{D^\times}(\{\alpha \in \mathcal{O}_D - \pi\mathcal{O}_D : \operatorname{tr}(\alpha) \equiv 0 \pmod{\pi^j\mathcal{O}_\infty}\})$$

where $\mu_{D^\times}$ is a fixed Haar measure of $D^\times$. From Lemma 7.1, we deduce that $|C_j|/|G_j| \ll 1/q^{d_\infty j}$.

We have a short exact sequence of groups:

$$1 \to \mathcal{O}_D^\times/(\mathcal{O}_\infty^\times(1+\pi^j\mathcal{O}_D)) \to G_j \xrightarrow{v} \mathbb{Z}/n\mathbb{Z} \to 0.$$

The group $\mathcal{O}_D^\times/(\mathcal{O}_\infty^\times(1+\pi^j\mathcal{O}_D))$ is isomorphic to $(\mathcal{O}_D/\pi^j\mathcal{O}_D)^\times/(\mathcal{O}_\infty/\pi^j\mathcal{O}_\infty)^\times$, and hence has cardinality

$$\frac{(q^{d_\infty n^2}-1)q^{d_\infty n^2\cdot(j-1)}}{(q^{d_\infty}-1)q^{d_\infty(j-1)}} = q^{d_\infty(n^2-1)j} \cdot \frac{1-1/q^{d_\infty n^2}}{1-1/q^{d_\infty}}.$$

This proves that there are positive constants c_1 and c_2, not depending on j, such that $c_1 q^{d_\infty(n^2-1)j} \le |G_j| \le c_2 q^{d_\infty(n^2-1)j}$. The required upper bound for $|C_j|$ follows from our bounds of $|C_j|/|G_j|$ and $|G_j|$. ☐

By Theorem 1.1, the index $[G_j : \widetilde{G}_j]$ can be bounded independent of j. Lemma 8.2 and the inclusion $\widetilde{C}_j \subseteq C_j$ shows that $|\widetilde{C}_j|/|\widetilde{G}_j| \ll 1/q^{d_\infty j}$ and $|\widetilde{C}_j|^{1/2} \ll q^{d_\infty(n^2-2)j/2}$.

We define L_j to be the fixed field in L^{sep} of the kernel of $\overline{\rho}_j$. Let $\mathcal{C}$ and $\mathcal{C}_j$ be smooth projective curves with function fields L and L_j, respectively. We can take $m_j := [\mathbb{F}_{L_j} : \mathbb{F}_L]$ above, where $\mathbb{F}_{L_j}$ and $\mathbb{F}_L$ are the field of constants of L_j and L, respectively. Theorem 1.1 implies that $\rho_\infty(\operatorname{Gal}(L^{\mathrm{sep}}/L\overline{k}))$ is an open subgroup of $\mathcal{O}_D^\times$, and hence $m_j \le [G_j : \overline{\rho}_j(\operatorname{Gal}(L^{\mathrm{sep}}/L\overline{k}))]$ can be bounded independently of j.

We define $|\mathscr{D}| := \sum_x \deg(x)$ where the sum is over the closed points of $\mathcal{C}$ for which the morphism $\mathcal{C}_j \to \mathcal{C}$, corresponding to the field extension L_j/L, is ramified. We may view X as an open subvariety of $\mathcal{C}$. Since the representation ρ_∞ is unramified at all closed points of X and $\mathcal{C} \setminus X$ is finite, we find that $|\mathscr{D}|$ can also be bounded independent of j.

Let $\mathcal{D}_{L_j/L}$ be the different of the extension L_j/L; it is an effective divisor of $\mathcal{C}_j$ of the form $\sum_x \sum_y d(y/x) \cdot y$, where the first sum is over the closed points x of $\mathcal{C}$ and the second sum is over

the closed points y of $\mathcal{C}_j$ lying over x. We define ϱ_j to be the smallest non-negative integer for which the inequality $d(y/x) \leq e(y/x)(\varrho_j + 1)$ always holds, where $e(y/x)$ is the usual ramification index. We will prove the following bound for ϱ_j in §8.1.

Lemma 8.3. *With notation as above, we have $\varrho_j \ll j+1$ where the implicit constant does not depend on j.*

Finally, we note that $\#|X|_d \ll q^{d_\infty d}/d$. For any integer $1 \leq j \leq \operatorname{ord}_\infty(a) + \lceil d/n \rceil$, combining all our bounds together we obtain

$$P_{\phi,a}(d) \ll \frac{1}{q^{d_\infty j}} \frac{q^{d_\infty d}}{d} + q^{d_\infty (n^2-2)j/2} \cdot j \cdot \frac{q^{d_\infty d/2}}{d} = \frac{q^{d_\infty (d-j)}}{d} + q^{d_\infty((n^2-2)j+d)/2} \cdot \frac{j}{d}$$

where the implicit constant depends only on ϕ. We choose $j := \operatorname{ord}_\infty(a) + \lceil d/n^2 \rceil$; for d sufficiently large, we do indeed have $1 \leq j \leq \operatorname{ord}_\infty(a) + \lceil d/n \rceil$. With this choice of j, we obtain the desired bound $P_{\phi,a}(d) \ll q^{d_\infty(1-1/n^2)d}$.

8.1. **Proof of Lemma 8.3.** Fix a non-constant $y \in A$ and define $h = -nd_\infty \operatorname{ord}_\infty(y) \geq 1$. Construct $\delta \in (L^{\mathrm{sep}})^\times$ and $a_0 = 1, a_1, a_2, \ldots \in L^{\mathrm{sep}}$ as in the beginning of §2. The series $u = \delta(\sum_{i=0}^\infty a_i \tau^{-i})$ then satisfies $u^{-1}\phi(F_\infty)u \subseteq \bar{k}(\!(\tau^{-1})\!)$. We noted above that $[\mathbb{F}_{L_j} : \mathbb{F}_L]$ can be bounded independently of j. So there is a finite subfield $\mathbb{F}$ of $\bar{k}$ that contains all the fields $\mathbb{F}_{L_j}$ and also the field with cardinality q^h. Set $K_0 = L\mathbb{F}(\delta)$, and recursively define the subfields $K_{i+1} := K_i(a_{i+1})$ of L^{sep} for $i \geq 0$. For $\sigma \in \operatorname{Gal}(L^{\mathrm{sep}}/L\bar{k})$, we have $\rho_\infty(\sigma) \in 1 + \pi^j \mathcal{O}_D$ if only if $v(\rho_\infty(\sigma) - 1) = v(\sigma(u)u^{-1} - 1) = v(\sigma(u) - u)$ is greater than or equal to $v(\phi_\pi^j) = jn$. This implies that L_j is a subfield of K_{jn}.

Consider a chain of global function fields $F_1 \subseteq F_2 \subseteq F_3$ with valuations v_1, v_2, and v_3, respectively (so v_3 lies over v_2 and v_2 lies over v_1). We then have $d(v_3/v_1) = e(v_3/v_2)d(v_2/v_1) + d(v_3/v_2)$, equivalently

$$\frac{d(v_3/v_1)}{e(v_3/v_1)} = \frac{d(v_2/v_1)}{e(v_2/v_1)} + \frac{d(v_3/v_2)}{e(v_3/v_1)}, \tag{8.2}$$

where $d(v_j/v_i)$ is the degree of the different $\mathcal{D}_{F_j/F_i}$ at v_j and $e(v_j/v_i)$ is the usual ramification index. Fix an integer j, and take any place v of L and any place w of L_j lying over v. Since $L \subseteq L_j \subseteq K_{jn}$, we can choose a place w' of K_{jn} lying over w. Using (8.2), we have

$$\frac{d(w/v)}{e(w/v)} \leq \frac{d(w'/v)}{e(w'/v)}.$$

It thus suffices to prove that $d(w/v)/e(w/v) \ll j+1$ holds for every place v of L, $j \geq 0$, and place w of K_j lying over v. Fix a place v of L.

Lemma 8.4. *There is a constant $B \geq 0$ such that $\operatorname{ord}_v(a_i) \geq -B$ holds for all $i \geq 0$ and all valuations $\operatorname{ord}_v \colon L^{\mathrm{sep}} \to \mathbb{Q} \cup \{+\infty\}$ extending ord_v.*

Proof. From (2.1), we find that

$$\frac{1}{q^h} \operatorname{ord}_v(a_i^{q^h} - a_i) \geq -C + \min_{\substack{0 \leq j \leq h-1 \\ i+j-h \geq 0}} \frac{\operatorname{ord}_v(a_j)}{q^{h-j}} \tag{8.3}$$

holds for some constant $C \geq 0$. Define $B := C/(1-1/q)$.

We will proceed by induction on i. The lemma is trivial for $i = 0$, since $\operatorname{ord}_v(a_0) = 0$. Now take $i \geq 1$. If $\operatorname{ord}_v(a_i) \geq 0$, then we definitely have $\operatorname{ord}_v(a_i) \geq -B$. Suppose that $\operatorname{ord}_v(a_i) < 0$. Then the roots of (2.1) as a polynomial in a_i are $a_i + b$ with b in the subfield of $\bar{k}$ of cardinality q^h; we

have $\operatorname{ord}_v(a_i+b)=\operatorname{ord}_v(a_i)$ for all such b (since $\operatorname{ord}_v(a_i)<0$), so $\operatorname{ord}_v(a_i)=\operatorname{ord}_v(a_i^{q^h}-a_i)/q^h$. By (8.3) and our inductive hypothesis, we deduce that

$$\operatorname{ord}_v(a_i)\geq -C-B/q=-B(1-1/q)-B/q=-B. \qquad \square$$

Lemma 8.5. *For a fixed integer $j\geq 0$, let w_j and w_{j+1} be places of K_j and K_{j+1}, respectively, such that w_j lies over v and w_{j+1} lies over w_j. We then have*

$$d(w_{j+1}/w_j)\leq Ce(w_{j+1}/v)$$

where C is a non-negative constant that does not depend on j.

Proof. Choose an extension $\operatorname{ord}_v\colon L^{\mathrm{sep}}\to\mathbb{Z}\cup\{+\infty\}$ that corresponds to w_{j+1} when restricted K_{j+1}. We defined a_{j+1} to be a root of the polynomial $X^{q^h}-X+\beta_j$ with $\beta_j\in K_j$. Let k_h be the subfield (of K_0) of cardinality q^h. We have $X^{q^h}-X+\beta_j=\prod_{b\in k_h}(X-a_{j+1}+b)$, so for each $\sigma\in\operatorname{Gal}_{K_j}$, there is a unique $\gamma(\sigma)\in k_h$ such that $\sigma(a_{j+1})=a_{j+1}+\gamma(\sigma)$. Since $k_h\subseteq K_j$, we find that $\gamma\colon\operatorname{Gal}_{K_j}\to k_h$ is a homomorphism whose image we will denote by H. Define the additive polynomial $g(X):=\prod_{b\in H}(X+b)\in k_h[X]$. The minimal polynomial of a_{j+1} over K_j is thus

$$g(X-a_{j+1})=g(X)-g(a_{j+1})\in K_j[X],$$

and the extension K_{j+1}/K_j is Galois with Galois group H.

The extension K_{j+1}/K_j is a variant of the familiar Artin-Schreier extensions. If $\operatorname{ord}_v(g(a_{j+1}))\geq 0$, then K_{j+1}/K_j is unramified at w_j [Sti93, Prop. 3.7.10(c)], so $d(w_{j+1}/w_j)=0$ and the lemma is trivial. So we may suppose that $m:=-\operatorname{ord}_v(g(a_{j+1}))>0$ and that K_{j+1}/K_j is ramified at w_j. We then find that K_{j+1}/K_j is totally ramified at w_j and that $d(w_{j+1}/w_j)\leq(|H|-1)(m+1)e(w_j/v)$ (see [Sti93, Prop. 3.7.10(d)]; the factor $e(w_j/v)$ arises by how we normalized our valuation). Therefore, $d(w_{j+1}/w_j)\leq(m+1)e(w_{j+1}/v)$. It thus suffices to prove that $\operatorname{ord}_v(g(a_{j+1}))$ can be bounded from below by some constant not depending on j; this follows immediately from Lemma 8.4. $\square$

We finally prove that $d(w/v)/e(w/v)\ll j+1$ holds for every place v of L, $j\geq 0$, and place w of K_j lying over v. If the place v corresponds to one of the closed points of X, then we know that ρ_∞, and hence K_j, is unramified at v; so $d(w/v)/e(w/v)=0$. We may now fix v to be a one of the finite many places of L for which ρ_∞ is ramified.

Fix a positive constant C as in Lemma 8.5. After possibly increasing C, we may assume that $d(w_0/v)\leq Ce(w_0/v)$ holds for every place w_0 of K_0 lying over v. Take any places w_j of K_j for $j\geq 0$ such that w_{j+1} lies over w_j and w_0 lies over v. By (8.2) and Lemma 8.5, we have

$$\frac{d(w_{j+1}/v)}{e(w_{j+1}/v)}=\frac{d(w_j/v)}{e(w_j/v)}+\frac{d(w_{j+1}/w_j)}{e(w_{j+1}/v)}\leq\frac{d(w_j/v)}{e(w_j/v)}+C.$$

Since $d(w_0/v)/e(w_0/v)\leq C$ by our choice of C, it is now easy to show by induction on j that $d(w_j/v)/e(w_j/v)\leq C(j+1)$ holds for all $j\geq 0$.

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